

Géza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

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1. FOREWORD

Declaring laconically that Géza Freud was interested in orthogonal polynomials would be an understatement rivaled by proclaiming that the Buckeyes are just another Big Ten football team or Mercedes is just one of many means of transportation available for mankind. As a matter of fact, approximately 88 items out of Freud's 132 approximation theory-related publications deal with orthogonal polynomials in one or another (possibly somewhat loosely defined) sense, and at least 35 of those have their primary 1980 AMS(MOS) Subject Classification given by 42C05. It is much more than symbolic that the first ("Remainder Term in a Tauberian Theorem, 1") and last ("On the Greatest Zero of an Orthogonal Polynomial") published papers by Freud (cf. items [Freud 1] and [Freud 131] in Freud's publication list in Volume 46 (January 1986) of this Journal or [Fr 1] and [Fr 71] in the references for this paper) do not just apply, discuss, treat, and review orthogonal polynomials but also contain the seeds of what I call Freud's seminal idea and contribution to the general theory of orthogonal polynomials. Perhaps nobody would argue that Freud was an orthogonal polynomialist in his heart even though he made extensive contributions to all of approximation theory including general, constructive, polynomial, rational and spline approximations, interpolation and harmonic analysis. It is much less known, however, that Freud had a Christoffel function syndrome (or fetish if you prefer), and this is what I classify as his fundamental gift to orthogonal polynomials, approximation theory, mathematics, and last but not least to my own mathematics in which Christoffel functions have been nourished and applied to a variety of problems. The rest of this paper in one or another sense is an elaboration of this idea and justification of my claim as to the significance of Christoffel functions as perceived and perfected by Géza Freud.

2. THE THESIS

It was Freud who first truly understood the fundamental significance of Christoffel functions, the way they permeate into various aspects of orthogonal polynomials; he was the first to apply, utilize and exploit them consciously to a variety of problems arising in orthogonal polynomials, approximation theory and harmonic and numerical analysis. His efforts resulted in (i) constructive and quantitative one-sided approximation by polynomials leading to Tauberian theorems with remainder terms; (ii) demonstrating strong Cesàro summability of orthogonal Fourier expansions of square integrable functions which eventually led to the formation

of a new theory of weighted approximations on the whole real line; (iii) improved asymptotics for orthogonal polynomials in the Szegő class; (iv) proving deep and substantial convergence results for orthogonal Fourier series, Hermite–Fejér and Lagrange interpolation processes and Gauss–Jacobi quadrature sums; and (iv) initiating the development of a general theory of orthogonal polynomials associated with measures on infinite intervals. Needless to say, the above five subjects are wholly interrelated and thus cannot be discussed and analyzed independently of each other.

One should be careful to avoid creating the false impression that, in fact, it was Freud's and only Freud's work that was of crucial consequence in the above-mentioned areas. As a matter of fact, it was not even in Freud's research that Christoffel functions first were shown to be so significant. Apart from earlier work by P. L. Chebyshev, C. F. Gauss, C. G. J. Jacobi, A. A. Markov, K. A. Posse, and T. J. Stieltjes on quadratures and the moment problem, one can find frequent use of Christoffel functions in work related to the uniqueness of the solution of the moment problem by N. I. Akhiezer, T. Carleman, H. Hamburger, M. G. Krein, and M. Riesz. Additional names and references will be mentioned at appropriate places in this paper.

What distinguishes Freud from his predecessors is the *systematic* and *consistent* nature of his efforts to put Christoffel functions to work for the benefit of approximation theory and orthogonal polynomials.

Even the latter claim needs some clarification and explanation. Namely, any carefully conducted study of Freud's mathematical thinking and creative procedures will undoubtedly reveal that he was driven towards Christoffel functions under the influence of P. Erdős and P. Turán, whose series of papers [ErTu1]–[ErTu3] bear primary responsibility for Freud's mathematical heritage.

3. NOTATIONS

Let $d\alpha$ be a finite positive Borel measure on the real line such that its support, $\text{supp}(d\alpha)$, is an infinite set, and all its moments, μ_n are finite, i.e.,

$$\mu_n = \int_{\mathbb{R}} t^n d\alpha(t) < \infty, \quad n = 0, 1, 2, \dots$$

Then there is a unique system $\{p_n\}$, $n = 0, 1, 2, \dots$, of polynomials orthonormal with respect to $d\alpha$ on the real line, i.e., polynomials

$$p_n(x) = p_n(d\alpha, x) = \gamma_n x^n + \dots, \quad \gamma_n = \gamma_n(d\alpha) > 0, \quad (3.1)$$

such that

$$\int_{\mathbb{R}} p_m(t) p_n(t) d\alpha(t) = \delta_{mn}, \quad m, n = 0, 1, 2, \dots \quad (3.2)$$

Define the Christoffel functions associated with $d\alpha$ by

$$\lambda_n(d\alpha, x) = \left[\sum_{k=0}^{n-1} |p_k(d\alpha, x)|^2 \right]^{-1}, \quad n = 1, 2, \dots \quad (3.3)$$

The Christoffel function $\lambda_n(d\alpha)$ is closely related to the Cotes numbers $\lambda_{kn} = \lambda_{kn}(d\alpha)$, $k = 1, 2, \dots, n$, which appear in the Gauss–Jacobi quadrature formula

$$\sum_{k=1}^n \Pi(x_{kn}) \lambda_{kn} = \int_{\mathbb{R}} \Pi(x) d\alpha(x) \quad (3.4)$$

valid for all polynomials Π of degree at most $2n - 1$. Here and hereafter $x_{kn} = x_{kn}(d\alpha)$, $k = 1, 2, \dots, n$, denote the zeros of $p_n(d\alpha)$ ordered by

$$x_{1n} > x_{2n} > \dots > x_{nn}. \quad (3.5)$$

The connection between the Christoffel function $\lambda_n(d\alpha)$ and the Cotes numbers $\lambda_{kn}(d\alpha)$ is given by

$$\lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn}(d\alpha)). \quad (3.6)$$

We write the three-term recurrence formula satisfied by the orthogonal polynomials (3.1) in the form

$$xp_n(d\alpha, x) = a_{n+1} p_{n+1}(d\alpha, x) + b_n p_n(d\alpha, x) + a_n p_{n-1}(d\alpha, x), \quad (3.7)$$

$n = 0, 1, \dots$, where $a_n = a_n(d\alpha)$ and $b_n = b_n(d\alpha)$ are given by

$$a_n = \gamma_{n-1}/\gamma_n \quad \text{and} \quad b_n = \int_{\mathbb{R}} t p_n(t)^2 d\alpha(t). \quad (3.8)$$

We will also need suitable notation to discuss orthogonal Fourier series and Lagrange interpolation. For $f \in L_1(d\alpha)$, its orthogonal Fourier series $S(d\alpha, f)$ in the orthogonal polynomials $p_k(d\alpha)$ is written as

$$S(d\alpha, f) = \sum_{k=0}^{\infty} c_k p_k(d\alpha). \quad (3.9)$$

The n th partial sum of its Fourier series is

$$S_n(d\alpha, f) = \sum_{k=0}^{n-1} c_k p_k(d\alpha) \quad (3.10)$$

and the Fourier coefficients $c_k = c_k(d\alpha, f)$ are given by

$$c_k = \int_{\mathbb{R}} f(t) p_k(d\alpha, t) d\alpha(t). \quad (3.11)$$

Let us define the reproducing kernel function $K_n = K_n(d\alpha)$ by

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x) p_k(d\alpha, t), \quad (3.12)$$

which, by the Christoffel–Darboux formula, can be written as

$$K_n(d\alpha, x, t) = \frac{\gamma_{n-1} p_n(d\alpha, x) p_{n-1}(d\alpha, t) - p_{n-1}(d\alpha, x) p_n(d\alpha, t)}{\gamma_n (x - t)}. \quad (3.13)$$

In terms of K_n , formula (3.10) takes the form

$$S_n(d\alpha, f, x) = \int_{\mathbb{R}} f(t) K_n(d\alpha, x, t) d\alpha(t). \quad (3.14)$$

The Lagrange interpolating polynomial $L_n(f) = L_n(d\alpha, f)$ associated with the function f is defined as the unique algebraic polynomial of degree at most $n - 1$ which agrees with f at the zeros of $p_n(d\alpha)$; it can be represented as

$$L_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) l_{kn}(d\alpha, x), \quad (3.15)$$

where the fundamental polynomials of Lagrange interpolation $l_{kn}(d\alpha)$ are defined by

$$l_{kn}(d\alpha, x) = \frac{p_n(d\alpha, x)}{p'_n(d\alpha, x_{kn})(x - x_{kn})}. \quad (3.16)$$

In this paper we also consider orthogonal polynomials on the unit circle. Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi]$ whose support is an infinite set. Then there is a unique system $\{\varphi_n\}$, $n = 0, 1, 2, \dots$, of polynomials orthonormal with respect to $d\mu$ on the unit circle, i.e., polynomials

$$\varphi_n(z) = \varphi_n(d\mu, z) = \kappa_n z^n + \dots, \quad \kappa_n = \kappa_n(d\mu) > 0, \quad (3.17)$$

such that

$$(2\pi)^{-1} \int_0^{2\pi} \varphi_m(d\mu, z) \overline{\varphi_n(d\mu, z)} d\mu(\theta) = \delta_{mn}, \quad z = e^{i\theta}, \quad m, n = 0, 1, 2, \dots \quad (3.18)$$

For orthogonal polynomials on the unit circle we define the Christoffel functions $\omega_n(d\mu)$ associated with $d\mu$ by

$$\omega_n(d\mu, z) = \left[\sum_{k=0}^{n-1} |\varphi_k(d\mu, z)|^2 \right]^{-1}, \quad n = 1, 2, \dots \quad (3.19)$$

In analogy with the real case, define the reproducing kernel function $K_n = K_n(d\mu)$ by

$$K_n(d\mu, z, u) = \sum_{k=0}^{n-1} \overline{\varphi_k(d\mu, z)} \varphi_k(d\mu, u). \quad (3.20)$$

It was proved by G. Szegő (cf. [Fr31b, p. 196]) that the analogue of the Christoffel–Darboux formula (3.13) is

$$K_n(d\mu, z, u) = \frac{\overline{\varphi_n^*(d\mu, z)} \varphi_n^*(d\mu, u) - \overline{\varphi_n(d\mu, z)} \varphi_n(d\mu, u)}{1 - \bar{z}u}. \quad (3.21)$$

Here and in what follows, the $*$ -transform of an n th-degree polynomial Π is defined by

$$\Pi^*(z) = z^n \bar{\Pi}(1/z), \quad (3.22)$$

where the conjugation refers to taking the complex conjugates of the coefficients of the polynomial Π . The monic orthogonal polynomials

$$\Phi_n(d\mu, z) = \kappa_n^{-1} \varphi_n(d\mu, z) \quad (3.23)$$

satisfy the recurrence formula

$$\Phi_{n+1}(d\mu, z) = z\Phi_n(d\mu, z) + \Phi_n(d\mu, 0) \Phi_n^*(d\mu, z), \quad n = 0, 1, \dots, \quad (3.24)$$

which turns out to be of fundamental significance in many problems related to orthogonal polynomials on the unit circle (cf. [Sz2, p. 293]).

If g is a nonnegative measurable function in $[0, 2\pi]$ such that $\log g \in L_1$, then the Szegő function $D(g)$ is defined by

$$D(g, z) = \exp \left\{ (4\pi)^{-1} \int_0^{2\pi} \log g(t) \frac{u+z}{u-z} dt \right\}, \quad u = e^{it}, |z| < 1. \quad (3.25)$$

Note that $D(g, 0)$ can be defined even when $\log g$ is not integrable. Of course, if $g \in L_1$ then $D(g, 0)$ does not vanish if and only if $\log g \in L_1$. Moreover, if $\log g \in L_1$, then $D(g) \in H_2$ in the unit disk, $D(g, z) \neq 0$ for $|z| < 1$, $D(g, 0) > 0$,

$$\lim_{r \nearrow 1} D(g, re^{it}) = D(g, e^{it}) \quad (3.26)$$

exists for almost every t in $[0, 2\pi]$ and

$$|D(g, e^{it})|^2 = g(t) \quad (3.27)$$

almost everywhere (cf. [Fr31b, Chap. 5; Sz2, Chap. 10]).

The symbol \sim , as in $A \sim B$ where A and B depend on some parameters, is used to indicate that $|A/B|$ and $|B/A|$ are both bounded uniformly in the given range of parameters.

The set of all algebraic polynomials of degree at most n is denoted by \mathbb{P}_n . The symbols \mathbb{R} and \mathbb{N} are used to denote the set of real numbers and positive integers, respectively.

4. JUSTIFICATION OF THE CLAIM

4.1. *A Little Philosophy*

The crux of the matter is the formula

$$\lambda_n(d\alpha, x) = \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(x) = 1}} \int_{\mathbb{R}} |\Pi(t)|^2 d\alpha(t). \quad (4.1.1)$$

Let us verbalize some of the obvious consequences of (4.1.1). First of all, the Christoffel function is a monotonic function of the measure, and thus, information regarding Christoffel functions of majorizing measures immediately yields similar information on Christoffel functions under consideration. The other, equally evident fact is that a quantity originating from orthogonal polynomials, that is, the reciprocal of the sum of the squares of the moduli of orthogonal polynomials, is, in fact, equivalent to a purely approximation theoretic quantity arising from best $L_2(d\alpha)$ approximations, and thus finding the Christoffel function asymptotically can be achieved by nearly optimal $L_2(d\alpha)$ approximations, which in practice boils down to finding suitable polynomials Π to substitute in the integral in (4.1.1). Formula (4.1.1) has been well known for many years, and its applications can be found in papers by P. Erdős, J. Shohat, and P. Turán (cf. [ErTu1–ErTu3, Sho4, Sho6, Sho8]), whose influence on Freud's research should not be overlooked.

For Christoffel functions associated with polynomials orthogonal on the unit circle, the formula analogous to (4.1.1) is given by

$$\omega_n(d\mu, z) = \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(z) = 1}} (2\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}. \quad (4.1.2)$$

The point is that the Christoffel function and the corresponding minimizing polynomial in (4.1.1) of some measures are well known. As a matter of fact, the extremal polynomial $\Pi = \Pi_x(d\alpha)$ in (4.1.1) is always given by

$$\Pi(t) = K_n(d\alpha, x, t)/K_n(d\alpha, x, x). \quad (4.1.3)$$

For example, if $d\alpha$ is the Chebyshev measure, that is, $d\alpha(t) = v dt$ where

$$v(t) = (1 - t^2)^{-1/2} (|t| < 1) \quad \text{and} \quad v(t) = 0 (|t| \geq 1), \quad (4.1.4)$$

then

$$\lambda_n(v, x)^{-1} = \pi^{-1} [n - \frac{1}{2} + U_{2n-2}(x)/2], \quad (4.1.5)$$

where U_n is the Chebyshev polynomial of the second kind, and the corresponding minimizing polynomial Π in (4.1.2) has an equally simple form (cf. [Fr31b, p. 104]).

PART 1: ORTHOGONAL POLYNOMIALS ON FINITE INTERVALS AND ON THE UNIT CIRCLE

4.2. *One-Sided Approximations and Tauberian Theorems with Remainder Terms*

On the basis of the extremal property (4.1.1) and formula (4.1.5), it becomes a matter of straightforward and routine calculations using standard techniques of approximation theory (cf. [Fr31b, Sect. 3.3, pp. 100–105]) to show that if w is defined by

$$w(x) = (-\log x)^{a-1}, \quad 0 < x < 1, a > 0, \quad (4.2.1)$$

then

$$\lambda_n(w, x) = O(1/n) \quad (4.2.2)$$

uniformly in $[0, 1]$. For given $x \in [0, 1]$, let Γ_x be defined by

$$\Gamma_x(t) = 1 \quad \text{for } 0 \leq t < x \quad \text{and} \quad \Gamma_x(t) = 0 \quad \text{for } x \leq t \leq 1. \quad (4.2.3)$$

For given n , the well-known construction of A. A. Markov and T. J. Stieltjes provides two polynomials, r and R , of degree at most $2n - 2$ such that

$$r(t) \leq \Gamma_x(t) \leq R(t), \quad 0 \leq t \leq 1, \quad (4.2.4)$$

and

$$\int_0^1 [R(t) - r(t)] w(t) dt = \lambda_n(w, x) \quad (4.2.5)$$

(cf. [Fr31b, p. 27]), and by (4.2.2) we obtain that the rate of one-sided $L_1(w)$ approximation of Heaviside's function Γ is $O(1/n)$.

The rest is history, and what I have sketched is how Freud obtained his first Tauberian theorem with remainder term in [Fr1]. The remaining ingredients of this Tauberian theorem come from S. N. Bernstein [Be1] (estimating coefficients of polynomials in terms of their L_1 -norm) and J. Karamata [Ka], whose ingenious one-sided approximation arguments in simplifying G. H. Hardy and J. E. Littlewood's proof of an improvement of Littlewood's Tauberian theorem [Li] are by now classical. Here is Freud's result.

THEOREM 4.2.1 [Fr1]. *Let τ be a nondecreasing function on \mathbb{R}^+ , and let f be a nonnegative function on \mathbb{R}^+ . Assume that the Lebesgue–Stieltjes integral*

$$F(s) = \int_{\mathbb{R}^+} f(t) \exp(-st) d\tau(t) \quad (4.2.6)$$

converges for all $s > 0$, and that there exists $a > 0$ such that

$$F(s) = K\Gamma(a+1) s^{-a} [1 + r(s)], \quad s > 0, \quad (4.2.7)$$

where $r(s)$ satisfies $|r(s)| < R(s)$ with $R \nearrow$, $R(0) = 0$ and $R(qs) \leq \exp(cq) R(s)$, c independent of q and s . Then, for every $b > 0$,

$$\int_0^x t^b f(t) d\tau(t) = Ka(a+b)^{-1} x^{a+b} [1 + O(|\log R(1/x)|^{-1})] \quad (4.2.8)$$

as $x \rightarrow \infty$.

It is interesting to point out that, independently of Freud, two other mathematicians (J. Korevaar [Kor] and A. G. Postnikov [Pos1]) published results of a similar nature, and though their approach was also via Karamata's method, their results were somewhat weaker than Freud's. More about Tauberian theorems is discussed by T. Ganelius in [Gan3], and one-sided approximation is touched upon by R. DeVore in his survey [De]. Freud himself returned to both Tauberian theorems and one-sided approximations in later papers (cf. [Fr4, Fr8, Fr10, Fr13, Fr14, FrGa, FrSz1, FrSz2, FrNe1, FrNe2, Fr50, Fr57, Fr58]). There are also two monographs by T. Ganelius [Gar2] and A. G. Postnikov [Pos3] dealing with Freud's results and related topics.

4.3. Convergence and Absolute Convergence of Orthogonal Fourier Series and Lebesgue Functions

Let us pass on to the next topic, which consists of the role played by Christoffel functions in the theory of convergence of orthogonal Fourier series. Our first example deals with Christoffel functions and Lebesgue functions. The latter are defined as the norms of the partial sum operators $S_n(d\alpha)$ considered as mappings from one space of functions to another. If, for instance, $\text{supp}(d\alpha)$ is compact, say, a subset of $[-1, 1]$, then it is convenient to define the Lebesgue function $\Omega_n(d\alpha, x)$ by

$$\Omega_n(d\alpha, x) = \sup_{\|f\|_C \leq 1} |S_n(d\alpha, f, x)|, \quad (4.3.1)$$

where $C = C[-1, 1]$. Lebesgue constants are defined as greatest values of Lebesgue functions over a suitable domain. First applying Schwarz' inequality and then Bessel's inequality to $S_n(d\alpha, f)$ in (3.10), one immediately obtains

$$\Omega_n(d\alpha, x) \leq \lambda_n(d\alpha, x)^{-1/2} \{\alpha[-1, 1]\}^{1/2}, \quad (4.3.2)$$

and thus the Christoffel functions fundamental property (4.1.1) can again be used to estimate Lebesgue functions, which, via Lebesgue's inequality, yields convergence results for orthogonal Fourier series of continuous functions. For instance, if $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha'(x) \geq \text{const } v(x)$, where v is the Chebyshev weight function (cf. (4.1.4)), then

$$\Omega_n(d\alpha, x) = O(n^{1/2}) \quad (4.3.3)$$

uniformly in $[-1, 1]$, and hence the corresponding orthogonal Fourier series converges uniformly in $[-1, 1]$ for all $\text{lip } \frac{1}{2}$ functions.

This standard argument has frequently been used by Erdős, Freud, Natanson, Shohat, Turán, and others (cf. [ErTu1–ErTu3; Fr31b, Chaps. III–IV; Nat; Sho4; Sho6; Sho8]). While there have been numerous attempts to improve (4.3.3) under fairly restrictive conditions on the measure $d\alpha$, and in particular, the estimate $O(n^{1/2})$ has been pushed down to $O(\log n)$ by several authors (cf. [Al1, Al2, Fr31b, Szö]), nevertheless, the first nontrivial improvement of the Lebesgue function estimate (4.3.3) under sufficiently general conditions was not achieved until 1976 when I succeeded in replacing O in (4.3.3) by o . The result I am referring to is buried in the apocalyptic [Ne19, Theorems 8.8., 8.9, p. 152], and its improvement, below, has not been published before.

THEOREM 4.3.1 (Nevai). *Assume $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha'(x) > 0$ almost everywhere in $[-1, 1]$. If α is continuous at $x \in [-1, 1]$, then*

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \Omega_n(d\alpha, x)^2 = 0. \quad (4.3.4)$$

If α is uniformly continuous on a closed set $\mathcal{M} \subset (-1, 1)$, then (4.3.4) is satisfied uniformly for $x \in \mathcal{M}$. If, in addition, $\log \alpha'(\cos \theta) \in L_1$, then

$$\lim_{n \rightarrow \infty} n^{-1/2} \Omega_n(d\alpha, x) = 0 \tag{4.3.5}$$

almost everywhere in $[-1, 1]$. Finally, if α is continuous and positive on an interval $\Delta \subset [-1, 1]$, then (4.3.5) holds uniformly on every closed subinterval of Δ .

No matter how innocent Theorem 4.3.1 looks, it is as deep as anything known at present on orthogonal polynomials. I will show later how the proof consists of putting together a few building blocks created by the younger generation (A. Maté, E. A. Rahmanov, V. Totik, and I) whose depth surpasses everything previously known in the general theory of orthogonal polynomials. What prevents me from proving this theorem right now is my elaborate dialectic plan of creating suspense and expectations which must culminate at the right moment. This climatic event will take place in Section 4.14.

For absolute convergence of orthogonal Fourier series, Christoffel functions are also an indispensable tool. Following S. N. Bernstein's [Be2] arguments for proving absolute convergence of trigonometric Fourier series of $\text{Lip } \varepsilon$ ($\varepsilon > \frac{1}{2}$) functions, one realizes that besides the requirements regarding the function whose orthogonal Fourier series is under consideration, the other ingredient is the assumption that

$$[\lambda_n(d\alpha, x)]^{-1} = O(1). \tag{4.3.6}$$

Indeed, what one does is divide the orthogonal Fourier series into diadic blocks, and then, by Schwarz' inequality,

$$\begin{aligned} \sum_{k=2^m+1}^{2^{m+1}} |c_k p_k(d\alpha)| &\leq \left\{ \sum_{k=2^m+1}^{2^{m+1}} |c_k|^2 \sum_{k=2^m+1}^{2^{m+1}} |p_k(d\alpha, x)|^2 \right\}^{1/2} \\ &\leq \left\{ \sum_{k=2^m+1}^{\infty} |c_k|^2 \sum_{k=0}^{2^{m+1}} |p_k(d\alpha, x)|^2 \right\}^{1/2} \\ &= E_{2^m}(d\alpha, f, 2) [\lambda_{2^{m+1}+1}(d\alpha, x)]^{-1/2}, \end{aligned} \tag{4.3.7}$$

where $E_n(d\alpha, f, 2)$ denotes the best $L_2(d\alpha)$ approximation of the function f whose orthogonal Fourier coefficients are c_k . On the basis of (4.3.7) one can easily produce and prove a number of theorems regarding absolute convergence of orthogonal Fourier series (cf. Mityagin [Mit]). For instance, Freud established the following

THEOREM 4.3.2 [Fr5]. *Let $\text{supp}(d\alpha) \subset [-1, 1]$ and suppose that (4.3.6)*

holds uniformly on a set $\mathcal{M} \subset [-1, 1]$. Then the orthogonal Fourier series expansion of $f \in L_2(d\alpha)$ in $\{p_n(d\alpha)\}$ converges uniformly and absolutely on the set \mathcal{M} provided that

$$\sum_{k=1}^{\infty} E_k(d\alpha, f, 2) k^{-1/2} < \infty, \quad (4.3.8)$$

and in particular, if $f \in \text{Lip } \varepsilon$ ($\varepsilon > \frac{1}{2}$).

If the reader starts to think that one of the messages I try to convey is that, in many problems of orthogonal polynomials, the boundedness of the orthogonal polynomial system (which is self-evident for the trigonometric system) can be replaced by boundedness in the sense of arithmetic means (Cesàro boundedness), then my efforts and intentions are well understood. I expect this to become even more convincing in the next section.

4.4 Strong Cesàro Summability of Orthogonal Series

This section is devoted to questions regarding Cesàro summability of orthogonal series. It occupies a central position in Freud's private universe and it also yields a process for nearly best approximation which is of crucial importance in approximation theory, in particular in the theory of weighted polynomial approximations. Throughout this section we deal with measures $d\alpha$ whose support is compact; without loss of generality one can assume that it lies in $[-1, 1]$.

Strong $(C, 1)$ (i.e., $|C, 1|$) summability of $S(d\alpha, f, x)$ is defined by requiring

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n |S_k(d\alpha, f, x) - f(x)| = 0. \quad (4.4.1)$$

It was G. Alexits who first suggested investigating $|C, 1|$ summability of general orthogonal Fourier series in orthogonal polynomials. The first significant achievement belongs to K. Tandori [Ta1, Ta2]. Tandori realized that T. Carleman's [Ca1] method of proving Hardy and Littlewood's [HarLi] theorem (which generalizes Lebesgue's theorem on almost everywhere convergence of Fejér means (i.e., $(C, 1)$ -means) of trigonometric Fourier series of integrable functions) can be adapted to the more general setting of orthogonal Fourier series in orthogonal polynomials. Tandori's success was due to the fact that the reproducing kernel function $K_n(d\alpha)$ of (3.12), similarly to the Dirichlet kernel function in trigonometric series, allows a closed representation in terms of the Christoffel–Darboux formula (3.13). However, Tandori's theorem on strong Cesàro summability of orthogonal Fourier series in [Ta1] makes

the assumption that the associated orthogonal polynomials are uniformly bounded in the interval where $|C, 1|$ summability is expected to hold. Thus, Tandori inadvertently struck out, whereas Freud's ingenious observation that Tandori's proof (or for that matter, the original one of Carleman) does not actually live and breathe on uniform boundedness of the orthogonal polynomial system, but in fact needs only Cesàro boundedness of the orthogonal polynomials (cf. (4.3.6)), gave Freud the walk of his lifetime, putting him on first base with a bright chance of a grand slam which did eventually materialize.

Let us take a close look at the way $|C, 1|$ sums can be estimated. Let us pick $f \in L_2(d\alpha)$ and $x \in (-1, 1)$, and let I_n and E_n be defined by

$$I_n = [-1, 1] \cap (x - 1/n, x + 1/n) \quad \text{and} \quad E_n = [-1, 1] \setminus I_n. \quad (4.4.2)$$

Let $k < n$. Then, by (3.14),

$$\begin{aligned} S_k(d\alpha, f, x) &= \int_{\mathbb{R}} f(t) K_k(d\alpha, x, t) d\alpha(t) \\ &= \int_{I_n} f(t) K_k(d\alpha, x, t) d\alpha(t) + \int_{E_n} f(t) K_k(d\alpha, x, t) d\alpha(t) \\ &= S_k^{(1)}(d\alpha, f, x) + S_k^{(2)}(d\alpha, f, x), \end{aligned} \quad (4.4.3)$$

and one estimates the latter two terms individually. By Schwarz' inequality,

$$\begin{aligned} |S_k^{(1)}(d\alpha, f, x)|^2 &\leq \int_{I_n} |K_k(d\alpha, x, t)|^2 d\alpha(t) \int_{I_n} |f(t)|^2 d\alpha(t) \\ &\leq \int_{\mathbb{R}} |K_k(d\alpha, x, t)|^2 d\alpha(t) \int_{I_n} |f(t)|^2 d\alpha(t) \\ &\leq \int_{\mathbb{R}} |K_n(d\alpha, x, t)|^2 d\alpha(t) \int_{I_n} |f(t)|^2 d\alpha(t) \\ &= \lambda_n(d\alpha, x)^{-1} \int_{I_n} |f(t)|^2 d\alpha(t), \end{aligned} \quad (4.4.4)$$

and thus

$$n^{-1} \sum_{k=1}^n |S_k^{(1)}(d\alpha, f, x)| \leq \left[\lambda_n(d\alpha, x)^{-1} \int_{I_n} |f(t)|^2 d\alpha(t) \right]^{1/2}. \quad (4.4.5)$$

Now we estimate $S_k^{(2)}$ in (4.4.3). For given f , n and x , let us define $F = F(f, n, x)$ by

$$F(t) = f(t)/(x - t) \text{ for } t \in E_n \quad \text{and} \quad F(t) = 0 \text{ for } t \in I_n. \quad (4.4.6)$$

Applying the Christoffel–Darboux formula (3.13) to (4.4.3), we obtain

$$S_k^{(2)}(d\alpha, f, x) = a_k [p_k(d\alpha, x) c_{k-1}(d\alpha, F) - p_{k-1}(d\alpha, x) c_k(d\alpha, F)], \quad (4.4.7)$$

where $a_k = \gamma_{k-1}/\gamma_k$ and $c_k(d\alpha, F)$ are the orthogonal Fourier coefficients of F (cf. (3.1), (3.8), and (3.11)). Since

$$a_k = \gamma_{k-1}/\gamma_k = \int_{-1}^1 t p_{k-1}(d\alpha, t) p_k(d\alpha, t) d\alpha(t) \leq 1, \quad (4.4.8)$$

we have

$$\sum_{k=1}^n |S_k^{(2)}(d\alpha, f, x)| \leq 2 \left[\sum_{k=0}^n |p_k(d\alpha, x)|^2 \right]^{1/2} \left[\sum_{k=0}^n |c_k(d\alpha, F)|^2 \right]^{1/2}, \quad (4.4.9)$$

and thus, by Bessel's inequality,

$$n^{-1} \sum_{k=1}^n |S_k^{(2)}(d\alpha, f, x)| \leq 2 [n\lambda_{n+1}(d\alpha, x)]^{1/2} \left[n^{-1} \int_{-1}^1 |F(t)|^2 d\alpha(t) \right]^{1/2} \quad (4.4.10)$$

The combination of inequalities (4.4.5) and (4.4.10) yields the desired estimate

$$\begin{aligned} n^{-1} \sum_{k=1}^n |S_k(d\alpha, f, x)| \\ \leq 2^{3/2} [n\lambda_{n+1}(d\alpha, x)]^{-1/2} \left[n \int_{I_n} |f(t)|^2 d\alpha(t) + n^{-1} \int_{-1}^1 |F(t)|^2 d\alpha(t) \right]^{1/2} \end{aligned} \quad (4.4.11)$$

(cf. (4.4.6) for the definition of F), which is the bread and butter of all results regarding strong Cesàro summability of orthogonal Fourier series in orthogonal polynomials. What remains to be done is to estimate the two integrals on the right-hand side of (4.4.11), and this can be accomplished via real analysis under various conditions on f and $d\alpha$ without further reference to orthogonal polynomials. As a matter of fact, these two integrals are identical (modulo $d\alpha$) to those which arise in Lebesgue's proof of his theorem on almost everywhere convergence of arithmetic means of trigonometric Fourier series, and thus one needs nothing but the notion of Lebesgue points of $d\alpha$ -integrable functions and the associated simple properties of such combined with the usual technique of integrating by parts applied to the second integral on the right-hand side of (4.4.11). In

what follows, I formulate two representative results by Freud and G. Alexits and D. Králik for measures with compact support.

THEOREM 4.4.1 [Fr2]. *Let $f \in L_2(d\alpha)$ and suppose that*

$$[n\lambda_n(d\alpha, x)]^{-1} = O(1) \quad (4.4.12)$$

on a set \mathcal{M} . Then, for (Lebesgue) almost every $x \in \mathcal{M}$, we have

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n |S_k(d\alpha, f, x) - f(x)| = 0. \quad (4.4.13)$$

THEOREM 4.4.2 [AIKr]. *Let f be bounded on an interval Δ containing the support of $d\alpha$, and let (4.4.12) be satisfied at some $x \in \Delta$. If, for $h \rightarrow 0$, we have $\alpha(x \times h) - \alpha(x) = O(|h|)$, then*

$$n^{-1} \sum_{k=1}^n |S_k(d\alpha, f, x)| \leq \text{const} \cdot \sup_{t \in \Delta} |f(t)|, \quad (4.4.14)$$

uniformly for $n = 1, 2, \dots$.

Naturally, every approximator's immediate reaction to (4.4.14) is that, then, the de la Vallée–Poussin sum

$$n^{-1} \sum_{k=n}^{2n-1} S_k(d\alpha, f, x) \quad (4.4.15)$$

converges to f with order $E_n(f)$ if f is continuous, where $E_n(f)$ is the best approximation of f by polynomials of degree at most $n - 1$, and this observation makes investigation of $|C, 1|$ sums so valuable in approximation theory.

Two problems arising at this point are conditions for the validity of (4.4.12) and the possibility of extending the results to measures with unbounded support.

4.5. Asymptotics for Christoffel Functions

Here the discussion is centered on estimating Christoffel functions. As in Section 4.4, we assume that $\text{supp}(d\alpha) \subset [-1, 1]$. I will not accompany the reader through the mazes leading to the right estimates. Providing historic perspective does not seem to be the right way of introducing the reader to the wonderful world of Christoffel functions. Instead, I will present the contemporary state of affairs immediately by formulating the following two results.

THEOREM 4.5.1 [MaNe1]. *If $\log \alpha'(\cos t) \in L_1$, then*

$$\begin{aligned} e^{-1}\pi\alpha'(x)(1-x^2)^{1/2} &\leq \liminf_{n \rightarrow \infty} n\lambda_n(d\alpha, x) \\ &\leq \limsup_{n \rightarrow \infty} n\lambda_n(d\alpha, x) = \pi\alpha'(x)(1-x^2)^{1/2} \end{aligned} \quad (4.5.1)$$

for almost every x in $[-1, 1]$.

THEOREM 4.5.2 [Ne19]. *Let $\log \alpha'(\cos t) \in L_1$, and let $\Delta \subset [-1, 1]$ be a given interval. If $1/\alpha' \in L_1[\Delta]$, then*

$$\lim_{n \rightarrow \infty} n\lambda_n(d\alpha, x) = \pi\alpha'(x)(1-x^2)^{1/2} \quad (4.5.2)$$

for almost every $x \in \Delta$. If $x \in (-1, 1)$, α is absolutely continuous in a neighborhood of x , and α' is continuous at x , then (4.5.2) holds. If α is absolutely continuous in a neighborhood of Δ and α' is continuous and positive in Δ , then (4.5.2) is satisfied uniformly for $x \in \Delta$.

Both of these theorems have their roots in the work of P. Erdős, G. Freud, G. Geronimus, J. Shohat, G. Szegő, and P. Turán (cf. [ErTu3; Fr11; Fr31a, b; Ger2; Sho4; Sho6; Sho8; Sz2; Sz4, Vol I, p. 437]), and I find it rather amusing that it was A. Máté and I who finally discovered and proved them. It is also worthwhile to point out that the proofs of both theorems contain essential ingredients missed by all of the above pioneers. In particular, prior to Theorem 4.5.1 the strongest result known regarding (C, 1) boundedness of orthogonal polynomials was the following theorem by G. Freud.

THEOREM 4.5.3 [Fr11]. *Let α' satisfy*

$$\int_0^\pi |\alpha'(\cos(t+h))/\alpha'(\cos t) - 1| dt = O(|\log |h||^{-a}) \quad (4.5.3)$$

as $h \rightarrow 0$, with some $a > 1$. Then

$$\liminf_{n \rightarrow \infty} n\lambda_n(d\alpha, x) > 0; \quad (4.5.4)$$

that is,

$$n^{-1} \sum_{k=0}^{n-1} |p_k(d\alpha, x)|^2 = O(1), \quad (4.5.5)$$

almost everywhere in $[-1, 1]$.

This theorem was reproduced by Freud in [Fr31a, b] and by Ya. L.

Geronimus in [Ger2]. Freud suspected that this condition (4.5.3) was somewhat superfluous, and was willing to believe that (4.5.5) should hold under the much less restrictive Szegő condition $\log \alpha'(\cos t) \in L_1$. Nevertheless, when in 1979 I showed him our Theorem 4.5.1 his first reaction was disbelief in the proof. As it turned out, Freud himself had tried very hard to prove it for about a quarter of a century, and thus he could not imagine that there is a relatively short and simple proof of this theorem. Although I will not and cannot give detailed proofs of these theorems in a few pages, I can still provide the reader with some clues and insight into the nature of the proofs.

One starts by introducing a sequence of positive operators $G_n(d\alpha)$ defined by

$$G_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \int_{\mathbb{R}} f(t) K_n(d\alpha, x, t)^2 d\alpha(t) \quad (4.5.6)$$

for $f \in L_1(d\alpha)$. These operators were thoroughly investigated in [Ne19, Chap. 6.2]. Because of the Christoffel–Darboux formula (3.13), $G_n(d\alpha)$ looks similar to Fejér’s sum (i.e., the arithmetic mean of partial sums) of trigonometric Fourier series. This similarity is much more than skin deep, and I succeeded in proving the following result in [Ne19, Theorem 6.2.27, p. 88].

THEOREM 4.5.4 [Ne19]. *Let $\log \alpha'(\cos t) \in L_1$, and let $\Delta \subset [-1, 1]$ be a given interval. Assume that $d\alpha$ is absolutely continuous and $\alpha' \in \text{Lip } \varepsilon$ ($\varepsilon > 0$) in a neighborhood of Δ . Let $f \in L_1(d\alpha)$ and suppose that f is bounded in $[-1, 1] \setminus \Delta$. Then*

$$\lim_{n \rightarrow \infty} G_n(d\alpha, f, x) = f(x) \quad (4.5.7)$$

almost everywhere in Δ . If, in addition, f is continuous at $x \in \Delta$, then (4.5.7) is satisfied, and if f is continuous in Δ , then (4.5.7) holds uniformly in Δ .

Theorem 4.5.4 itself is based on Szegő’s theory and its refinements discovered by Freud [Fr16] (cf. [Fr31a, b]; [Ger2]). Now the point is that for the measure in Theorem 4.5.4 one knows the asymptotic behavior of the Christoffel functions. This was found by Ya. L. Geronimus [Ger2, Theorem 5.7] and I formulate it as

THEOREM 4.5.5 [Ger2]. *Let $d\alpha$ and Δ satisfy the conditions of Theorem 4.5.4. Then*

$$\lim_{n \rightarrow \infty} n \lambda_n(d\alpha, x) = \pi \alpha'(x) (1 - x^2)^{1/2}, \quad (4.5.8)$$

uniformly for $x \in \Delta$.

The next ingredient for the proof of Theorem 4.5.2 is the following

THEOREM 4.5.6 [Ne19, p. 26]. *If $d\alpha$ is supported in $[-1, 1]$ and α' is positive almost everywhere there, then*

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) / \lambda_{n+1}(d\alpha, x) = 1 \quad (4.5.9)$$

and

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) p_n(d\alpha, x)^2 = 0 \quad (4.5.10)$$

for every $x \in [-1, 1]$. Moreover, (4.5.9) and (4.5.10) hold uniformly in every closed subinterval of $(-1, 1)$.

A weaker version of Theorem 4.5.6, where the condition $\alpha' > 0$ a.e. is replaced by $\log \alpha'(\cos t) \in L_1$ and no uniform convergence in (4.5.10) is claimed, was discovered by Geronimus [Ger2, Theorem 3.4]. The proof of Theorems 4.5.4 and 4.5.6 is based on the following theorem of E. A. Rahmanov, which I think is one of the fundamental results in creating generalizations of Szegő's theory of orthogonal polynomials.

THEOREM 4.5.7 [Rah4]. *If $\text{supp}(d\alpha) \subset [-1, 1]$ and $\alpha' > 0$ a.e. in $[-1, 1]$, then the recurrence coefficients a_n and b_n in the three-term recurrence formula (3.7) satisfy*

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = 0. \quad (4.5.11)$$

Theorem 4.5.7 was originally stated in [Rah1]. In [MáNe2] it was pointed out that the proof of (4.5.11) in [Rah1] contained a well-hidden error since it referred to a result by Ya. L. Geronimus in [Sz3, p. 376] (cf. [Ger9]) which itself contained a misprint. In reaction to our paper [MáNe2], Rahmanov gave a correct proof of (4.5.11) in [Rah4]. His proof is rather tedious and long, and in [MáNeTo2] we succeeded in giving a shorter proof that we believe is simpler and illuminates better the reasons that lie behind (4.5.11). Our proof is based on an important integral inequality of A. N. Kolmogorov concerning conjugate functions and on some simple identities involving orthogonal polynomials.

The last building block in the proof of Theorem 4.5.2 is the following proposition of mine which enables one to estimate ratios of Christoffel functions associated with different measures in term of the operators G_n defined in (4.5.6).

THEOREM 4.5.8 [Ne19]. *Let $d\alpha$ and $d\beta$ be two positive Borel measures*

(for which orthogonal polynomials exist) on the real line, not necessarily with compact support. Suppose that $d\alpha$ can be represented in terms of $d\beta$ as

$$d\alpha = g d\beta, \quad (4.5.12)$$

where $g(\geq 0) \in L_1(d\beta)$. Then, for every polynomial P of degree m ,

$$|P(x)|^2 \frac{\lambda_n(d\alpha, x)}{\lambda_{n-m}(d\beta, x)} \leq G_{n-m}(d\beta, g|P|^2, x), \quad n > m. \quad (4.5.13)$$

If R is a polynomial of degree M such that $R^2 g^{-1} \in L_1(d\beta)$, then

$$|R(x)|^2 G_{n+M}(d\beta, g^{-1}|R|^2, x)^{-1} \leq \frac{\lambda_n(d\alpha, x)}{\lambda_{n+M}(d\beta, x)}. \quad (4.5.14)$$

Proof of Theorem 4.5.8. This proof is so simple that I will reproduce it here. By (4.1.1),

$$\lambda_n(d\alpha, x) \leq |P(x)|^{-2} K_{n-m}(d\beta, x, x)^{-2} \int_{\mathbb{R}} |P(t)|^2 K_{n-m}(d\beta, x, t)^2 d\alpha(t), \quad (4.5.15)$$

and thus, by (3.12) and (4.5.6), inequality (4.5.13) follows. To prove (4.5.14) we pick an arbitrary polynomial Π of degree $n-1$. Then we have

$$\Pi(x) R(x) = \int_{\mathbb{R}} \Pi(t) R(t) K_{n+M}(d\beta, x, t) d\beta(t), \quad (4.5.16)$$

and applying Schwarz' inequality, we obtain

$$|\Pi(x) R(x)|^2 \leq \int_{\mathbb{R}} |\Pi(t)|^2 g(t) d\beta(t) \int_{\mathbb{R}} g(t)^{-1} |R(t)|^2 K_{n+M}(d\beta, x, t)^2 d\beta(t), \quad (4.5.17)$$

which implies (4.5.14). ■

Using Theorems 4.5.4–4.5.8, the proof of Theorem 4.5.2 can be accomplished in a few lines.

Proof of Theorem 4.5.2. Let $d\alpha$ and Δ satisfy the conditions of Theorem 4.5.2. Define $d\beta$ and $d\sigma$ by

$$d\beta(t) = d\alpha(t) \text{ on } [-1, 1] \setminus \Delta^* \quad \text{and} \quad d\beta(t) = dt \text{ on } \Delta^* \quad (4.5.18)$$

and

$$d\sigma(t) = d\alpha(t) \text{ on } [-1, 1) \setminus \Delta^* \quad \text{and} \quad d\sigma(t) = \alpha'(t) dt \text{ on } \Delta^*, \quad (4.5.19)$$

where \mathcal{A}^* is a sufficiently close neighborhood of \mathcal{A} . Then, by Theorem 4.5.5,

$$\lim_{n \rightarrow \infty} n\lambda_n(d\beta, x) = \pi\beta'(x)(1-x^2)^{1/2} \quad (4.5.20)$$

uniformly for $x \in \mathcal{A}$. Moreover,

$$d\sigma = g d\beta, \quad (4.5.21)$$

where

$$g(t) = 1 \text{ on } [-1, 1] \setminus \mathcal{A}^* \quad \text{and} \quad g(t) = \alpha'(t) \text{ on } \mathcal{A}^*. \quad (4.5.22)$$

Thus, by Theorems 4.5.4 and 4.5.8 and formula (4.5.20), the asymptotic formula

$$\lim_{n \rightarrow \infty} n\lambda_n(d\sigma, x) = \pi\sigma'(x)(1-x^2)^{1/2} \quad (4.5.23)$$

holds either almost everywhere or pointwise or uniformly, depending on the particular properties of $d\alpha$. If $d\sigma \neq d\alpha$, then passing from $d\sigma$ to $d\alpha$ is accomplished via (4.1.1), which makes it possible to compare the corresponding Christoffel functions. Namely, by (4.5.19),

$$d\sigma \leq d\alpha \quad (4.5.24)$$

so that, by (4.1.1),

$$\lambda_n(d\sigma, x) \leq \lambda_n(d\alpha, x), \quad (4.5.25)$$

and therefore, by (4.5.21),

$$\liminf_{n \rightarrow \infty} n\lambda_n(d\alpha, x) \geq \pi\sigma'(x)(1-x^2)^{1/2} = \pi\alpha'(x)(1-x^2)^{1/2} \quad (4.5.26)$$

for almost every $x \in \mathcal{A}$. What remains to be shown is that

$$\limsup_{n \rightarrow \infty} n\lambda_n(d\alpha, x) \leq \pi\alpha'(x)(1-x^2)^{1/2} \quad (4.5.27)$$

for almost every $x \in \mathcal{A}$ as well. Here again (4.1.1) helps us out. Let us define Π by

$$\Pi(t) = K_n(v, x, t)/K_n(v, x, x), \quad (4.5.28)$$

where v is the Chebyshev weight; that is,

$$v(t) = (1-t^2)^{-1/2} \quad (|t| < 1) \quad \text{and} \quad v(t) = 0 \quad (|t| \geq 1). \quad (4.5.29)$$

Then by (4.1.1)

$$n\lambda_n(d\alpha, x) \leq nK_n(v, x, x)^{-2} \int_{-1}^1 K_n(v, x, t)^2 d\alpha(t). \quad (4.5.30)$$

The explicit expression for Π in (4.5.28) is well known (cf. [Fr31b, p. 244]) and it is easy to see that

$$\Pi^2(t) = O(n) [1 + n^2(x - t)^2]^{-1}. \quad (4.5.31)$$

Consequently, the right side of (4.5.30) behaves exactly as the Fejér sums of trigonometric Fourier series of measures (cf. [Ne19, p. 31; Zy1, p. 105]). Now, following Lebesgue's arguments applied to (4.5.30), one immediately obtains (4.5.27). This completes the proof of Theorem 4.5.2. ■

The next step is the proof of Theorem 4.5.1. Before proceeding with the proof, I formulate the following

Conjecture 4.5.9. If $\log \alpha'(\cos t) \in L_1$, then

$$\lim_{n \rightarrow \infty} n\lambda_n(d\alpha, x) = \pi\alpha'(x)(1 - x^2)^{1/2} \quad (4.5.32)$$

for almost every x .

A proof of this conjecture would bring to a natural climax investigations which were started by Szegő approximately 70 years ago in connection with Hankel forms [Sz4, Vol. I, p. 53] and equiconvergence of orthogonal Fourier series [Sz4, Vol. I, p. 437].

While in the proof of Theorem 4.5.2 the expert eye can recognize traces of ideas originating with Erdős, Freud, Geronimus, and Turán (cf. [ErTu3, Fr19b, Ger2]), the proof of Theorem 4.5.1 is totally novel. This is not unexpected since, earlier, authors did not investigate Christoffel functions on the set of orthogonality under the sole condition that $\log \alpha'(\cos t) \in L_1$. As is frequently the case, if one is to prove a deep result for orthogonal polynomials on the real line, then first one has to make a temporary transition to the unit circle and work within the framework of Szegő's theory. The two main ingredients in the proof of Theorem 4.5.1 are the following two results. Throughout the rest of this section we deal with orthogonal polynomials, measures and Szegő functions on the unit circle (cf. formulas (3.17)–(3.27)).

THEOREM 4.5.10 [Sz2, p. 297]. *If $\log \mu' \in L_1$, then*

$$\lim_{n \rightarrow \infty} \varphi_n^*(d\mu, z) = D(\mu', z)^{-1} \quad (4.5.33)$$

uniformly on compact subsets of the open unit disk.

THEOREM 4.5.11 [MáNe1]. *Let P be a polynomial of degree n and let f be analytic in an open set containing the closed unit disk. Then*

$$|P(z)|^p |f(rz)| \leq (2 + np)e(8\pi)^{-1} \int_0^{2\pi} |P(u)|^p |f(u)| d\theta, \quad u = e^{i\theta} \quad (4.5.34)$$

for every positive p , where z is an arbitrary point with $|z| = 1$ and $r = np/(2 + np)$.

As a matter of fact, in the inequality given in [MáNe1, Theorem 6, p. 148], $|f|^2$ rather than $|f|$ appears, and thus it is apparently (but not actually) weaker than (4.5.34). The latter can be established directly, in a way similar to the way in which it was established in [MáNe1], by using the contour integral formula

$$(1 - r^2) g(r) = (2\pi i)^{-1} \int_{|\zeta|=1} g(\zeta)(1 - r\zeta)(\zeta - r)^{-1} d\zeta, \quad (4.5.35)$$

valid for g analytic in the closed unit disk, with $g(z) = (R_n(sz))^p f(sz)$ ($s \nearrow 1$); here R_n is a polynomial of degree n having no zeros inside the unit disk such that $|R_n(z)P_n(z)^{-1}| = 1$ for $|z| = 1$ (see [MáNe1, p. 149]). Note that $|(1 - r\zeta)(\zeta - r)^{-1}| = 1$ for $|\zeta| = 1$, which makes the estimation of the integral on the right-hand side of (4.5.35) easy.

Instead of proving Theorem 4.5.1 in its entirety, I prove only the statement regarding the limit inferior of the Christoffel function, and even that part will be done for orthogonal polynomials on the unit circle. The transition from the unit circle back to the interval $[-1, 1]$ is accomplished via the inequality

$$\lambda_n(d\alpha, x) \geq \pi \omega_n(d\mu, z), \quad x = \cos \theta, z = e^{i\theta} \quad (4.5.36)$$

(cf. [MáNe1, p. 152]), where $d\mu$ is the projection measure associated with $d\alpha$ by $d\mu(\theta) = d\alpha(\cos \theta)$.

THEOREM 4.5.12 [MáNe1]. *If $\log \mu' \in L_1$, then*

$$2e^{-1} \mu'(\theta) \leq \liminf_{n \rightarrow \infty} n \omega_n(d\mu, e^{i\theta}) \quad (4.5.37)$$

for almost every real θ .

Proof of Theorem 4.5.12. Let Π be a polynomial of degree n and let m be an arbitrary integer greater than n . Since $\varphi_m^*(d\mu)$ has no zeros in the closed unit disk [Fr31b, p. 198], we can apply inequality (4.5.34) with $p = 2$ to obtain

$$\begin{aligned}
& |\Pi(e^{i\theta})|^2 |\varphi_m^*(d\mu, re^{i\theta})|^{-2} \\
& \leq en(4\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 |\varphi_m^*(d\mu, u)|^{-2} dt, \quad u = e^{it}, \quad (4.5.38)
\end{aligned}$$

where $r = 1 - n^{-1}$. The first m moments of the measure $|\varphi_m^*(d\mu, e^{it})|^{-2} dt$ coincide with those of $d\mu(t)$ [Fr31b, p. 198]. Therefore, since $m > n$, we have

$$|\Pi(e^{i\theta})|^2 |\varphi_m^*(d\mu, re^{i\theta})|^{-2} \leq en(4\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}. \quad (4.5.39)$$

Now letting $m \rightarrow \infty$ and using (4.5.33), we get

$$2e^{-1} |D(\mu', re^{i\theta})|^2 \leq n |\Pi(e^{i\theta})|^{-2} (2\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}. \quad (4.5.40)$$

Since Π is an arbitrary polynomial of degree $n - 1$, we can conclude that

$$2e^{-1} |D(\mu', re^{i\theta})|^2 \leq n\omega_n(d\mu, e^{i\theta}), \quad (4.5.41)$$

where $r = n^{-1}$. Note that $\log |D(\mu', z)|^2$ is the Poisson integral of $\log \mu' \in L_1$. Hence

$$\lim_{p \nearrow 1} |D(\mu', pe^{i\theta})|^2 = \mu'(\theta) \quad (4.5.42)$$

for almost every real θ . Therefore, as $n \rightarrow \infty$ in (4.5.41), inequality (4.5.37) follows and so does Theorem 4.5.12. ■

I conclude this section by mentioning a generalization of Theorem 4.5.2 which I gave in [Ne19, Theorem 4.1.19, p. 37, and Corollary 6.2.53, p. 104]. It is one of my favorite results on pointwise convergence properties of Christoffel functions.

THEOREM 4.5.13 [Ne19]. *Let $m > 0$ be a fixed integer, and let $\Delta \subset [-1, 1]$ be a given interval. Let $\log \alpha'(\cos t) \in L_1$. If $1/\alpha' \in L_1[\Delta]$, then*

$$\begin{aligned}
& \lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} p_k(d\alpha, x) p_{k+m}(d\alpha, x) \\
& = T_m(x) [\pi \alpha'(x) (1 - x^2)^{1/2}]^{-1} \quad (4.5.43)
\end{aligned}$$

for almost every $x \in \Delta$, where $T_m(x) = \cos(mt)$, $x = \cos t$, denotes the m th Chebyshev polynomial of the first kind.

4.6. How Grenander and Rosenblatt and Geronimus Erred

In light of the relative freshness of the Christoffel function asymptotics given by Theorems 4.5.1 and 4.5.2, the reader may well ask whether results of comparable strength were available prior to 1976. The answer to this is a straightforward yes and no. More accurately, there were some results which are much more powerful than the above theorems, but unfortunately either their statements or their proofs are false. Over the years I have had the dubious honor of finding errors such as the one in [Rah1] (cf. [MáNe2]; and the two gems presented below form part of my valuable collection of goofs by mathematicians par excellence.

The first one belongs to U. Grenander and M. Rosenblatt [GrRo], who considered a generalization of the extremal problem (4.1.2) defining the Christoffel function on the unit circle. This generalization amounts to replacing the condition $\Pi(z)=1$ by prescribing the value of the polynomial Π and its derivatives of given orders at several points. First then find the explicit solution of this minimum problem in terms of determinants involving the kernel function K_n and its derivatives (cf. (3.20)) and they succeed in obtaining asymptotics for these generalized Christoffel functions when all the interpolation points are inside the open unit disk (cf. [GrRo, Theorem 1, p. 113]). Then they consider the case where the interpolation points are on the unit circle, and they formulate and “prove” a statement [GrRo, Theorem 2, p. 115] which seems to surpass Theorem 4.5.2 significantly in several respects. Here I limit myself to giving the following partial case of this statement which provides asymptotics for the Christoffel functions $\omega_n(d\mu, e^{i\theta})$.

Claim 4.6.1 [GrRo]. Let $d\mu$ be absolutely continuous and assume that μ' is positive and continuous. Then

$$n\omega_n(d\mu, e^{i\theta}) = \mu'(\theta) + O(1/n) \quad (4.6.1)$$

uniformly for all real θ .

When I first saw this paper I immediately knew that there was something wrong, and it did not take long for me to catch the error in the proof. However, it took me several years to convince myself that it was not just the proof but also the statement which was wrong. I hope that by now the reader agrees with me that a statement such as (4.6.1) cannot possibly be true without imposing extra conditions on μ' . Yes, my reader, you are right: the asymptotic formula (4.6.1) is actually stronger than a special case of Steklov’s conjecture, which I formulate as

$$\varphi_n(d\mu, z) = O(1), \quad n = 1, 2, \dots, \quad (4.6.2)$$

uniformly for all z with $|z| = 1$, whenever the measure $d\mu$ is absolutely continuous, μ' is continuous and $\mu'(\theta) \geq \text{const} > 0$. (The original Steklov conjecture claims (4.6.2) for all absolutely continuous measures $d\mu$ for which $\mu'(\theta) \geq \text{const} > 0$.) Due to E. A. Rahmanov's marvelous paper [Rah3], we know that Steklov's original conjecture is false, and thus Claim 4.6.1 of Grenander and Rosenblatt is not likely to be correct either.

Where does the proof of (4.6.1) fail? The authors try to follow the road paved by S. N. Bernstein and G. Szegö (cf. [Be3–Be5; Sz2, p. 31; Sz4, Vol. I, p. 69]) in that they first prove it when μ' is the reciprocal of a positive trigonometric polynomial, in which case it does indeed hold. Passing to the general case is accomplished by a one-sided approximation, $P < \mu' < R$, of μ' by the reciprocals P, R of positive trigonometric polynomials such that $|P - R| < \varepsilon$. The point is that by (4.1.2), $\omega_n(d\mu, e^{i\theta})$ is between the corresponding Christoffel functions of P and R . So far everything is fine. However, at this point the authors let $\varepsilon \rightarrow 0$ and claim to have completed the proof of the theorem (cf. [GrRo, p. 118, line 12 from below]). We all know that ε 's and $O(1/n)$'s do not mix well, and thus the last line of the proof nullifies everything.

Apart from the unfortunate Theorem 2, Grenander and Rosenblatt's paper [GrRo] does possess intrinsic value. Those who are familiar with my research know that some of my favorite ideas originated from this paper.

The other error was made by Geronimus in [Ger5] and repeated in [Ger6] (cf. [Su, p. 23]). In these papers Geronimus attempts to prove.

Claim 4.6.2 [Ger5]. The asymptotic formula

$$\lim_{n \rightarrow \infty} n\omega_n(d\mu, e^{i\theta}) = \mu'(\theta) \quad (4.6.3)$$

holds for almost every real θ , provided that $d\mu$ satisfies some extremely weak conditions; in particular, $\mu' > 0$, a.e. would suffice.

It is my wishful thinking that this theorem of Geronimus is actually correct, and I am in no position to prove otherwise. However, his proof also relies on the "fact" that the order of taking limits can be interchanged, and this is accomplished in a way which is very similar to the Grenander–Rosenblatt argument or, for that matter, to Cauchy's "proof" that the limit of a convergent sequence of continuous functions is continuous.

In his attempt to prove (4.6.3), Geronimus considers the zeros of $K_n(d\mu, z, z_0)$ where $z_0, |z_0| = 1$, is a fixed point (cf. (3.20)). As shown by Szegö [Sz2, p. 292], all such zeros have modulus 1. Then Geronimus uses arguments borrowed from P. P. Korovkin [Koro] and J. L. Walsh [Wa, Sects. 7.3–7.4] to show that the asymptotic distribution of these zeros on the unit circle is governed by a function called the Robin distribution

function associated with $d\mu$. (In case $\mu' > 0$ a.e., the Robin function is identically θ .) Afterwards Geronimus writes

$$n\omega_n(d\mu, e^{i\theta}) = [A_n(\theta + \varepsilon) - A_n(\theta - \varepsilon)]/[B_n(\theta + \varepsilon) - B_n(\theta - \varepsilon)], \quad (4.6.4)$$

where $\varepsilon > 0$ is sufficiently small, and actually depends on n . For fixed $\varepsilon > 0$, the expression $[A_n(\theta + \varepsilon) - A_n(\theta - \varepsilon)]/[B_n(\theta + \varepsilon) - B_n(\theta - \varepsilon)]$ converges as $n \rightarrow \infty$, and the proof uses the above-mentioned zero distribution properties. However, in (4.6.4), ε depends on n , and thus one cannot let $n \rightarrow \infty$ without making some additional assumptions on the measure. Nevertheless, Geronimus lets $n \rightarrow \infty$ in (4.6.4) (cf. [Ger5, p. 1388, formula (9); Ger6, p. 46, line 2 from below]).

The resulting falsely proved theorem was used by B. L. Golinskiĭ in [Gol1] to "prove"

Claim 4.6.3 [Gol1]. If $\mu' > 0$ almost everywhere, then

$$\omega_n(d\mu, e^{i\theta})^{-1} \int_{\theta - 1/(2n)}^{\theta + 1/(2n)} d\mu(t) \leq 8\pi, \quad n = 1, 2, \dots, \quad (4.6.5)$$

uniformly for all real θ .

For other true and/or false results related to Geronimus' [Ger5, Ger6], I refer the reader to P. K. Suetin's now obsolete survey paper [Su, pp. 22–26], where a number of theorems of this nature are given.

4.7. Quadrature Sums and Christoffel Functions

By the Gauss–Jacobi quadrature formula (3.4),

$$\sum_{k=1}^n |\Pi(x_{kn})|^2 \lambda_{kn} = \int_{\mathbb{R}} |\Pi(x)|^2 d\alpha(x) \quad (4.7.1)$$

(cf. (3.5) and (3.6)) for all real polynomials $\Pi(x)$ of degree at most $n - 1$. Naturally, we cannot expect to be able to extend (4.7.1) to

$$\sum_{k=1}^n |\Pi(x_{kn})|^p \lambda_{kn} = \int_{\mathbb{R}} |\Pi(x)|^p d\alpha(x) \quad (4.7.2)$$

for $p > 0$ except when $|\Pi|^{p/2}$ is a polynomial of degree at most $n - 1$. Fortunately, it turns out that it is not (4.7.2) which is needed in several problems related to orthogonal polynomials, quadratures and interpolation but rather the inequality

$$\sum_{k=1}^n |\Pi(x_{kn})|^p \lambda_{kn} \leq K \int_{\mathbb{R}} |\Pi(x)|^p d\alpha(x) \quad (4.7.3)$$

for all (or possible some) $p > 0$ and for all polynomials Π of degree m , where K is a constant depending on the measure, the exponent p and the ratio m/n only.

It was R. Askey [As4, As5] who realized the importance of inequalities of the type (4.7.3) when investigating weighted mean convergence of Lagrange interpolation at zeros of Jacobi polynomials. In [As3, p. 533], Askey posed the problem of proving (4.7.3) for various classes of measures. One can trace the origin of inequalities of type (4.7.3) to J. Marcinkiewicz [Mar], who used the analogue of such an inequality to prove the L_p convergence of trigonometric interpolation at equidistant points, for all $p > 0$.

While

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n |f(x_{kn})|^p \lambda_{kn} = \int_{\mathbb{R}} |f(x)|^p d\alpha(x) \quad (4.7.4)$$

for all continuous functions f when, say, the support of the measure is a compact set (cf. [Fr31b, p. 89]), it is obvious that the norm of the mapping $F: C^* \rightarrow \mathbb{R}_n^*$ defined by $F(f) = \{f(x_{kn})\}$, $k = 1, 2, \dots, n$, cannot be bounded unless the measure is a finite union of n mass points. Here C^* is the space of continuous functions defined on the shortest interval containing the support of the measure with norm defined by the p th root of the integral in (4.7.4), whereas \mathbb{R}_n^* is the n -dimensional space where the norm is defined by the p th root of the quadrature sum in (4.7.4). On the other hand, the existence of an inequality of type (4.7.3) indicates that F restricted to some finite-dimensional subspaces (i.e., polynomials of a suitable degree) is not merely bounded (which is obvious) but also uniformly bounded in n . An application of (4.7.3.) is discussed in Section 4.8, where I say more about Lagrange interpolation.

From 1974 through 1976, I worked on a number of problems related to weighted mean convergence of Lagrange interpolation taken at zeros of orthogonal polynomials, and one of the most resistant ones was Turán's problem which amounts to finding out whether there exist measures for which one cannot push the convergence of Lagrange interpolation $L_n(d\alpha, f)$ beyond $L_2(d\alpha)$. In solving Turán's problem, I encountered two problems. The first one is, in some sense, the dual of (4.7.3) and consists of finding lower bounds for

$$\sum_{k=1}^n |p_{n-1}(d\alpha, x_{kn})|^p \lambda_{kn}. \quad (4.7.5)$$

Clearly, when $p = 0$ and $p = 2$, the sum in (4.7.5) equals $\int d\alpha$ and 1, respectively. Whether one can interpolate between 0 and 2 remains to be seen. It is even more difficult to determine lower bounds for

$$\sum_{k \in I} |p_{n-1}(d\alpha, x_{kn})|^p \lambda_{kn}, \quad (4.7.6)$$

where \mathbf{I} is a given set of indices k . The other problem pertains to the continuous analogue of the previous one and requires determining lower bounds for

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}} |p_n(d\alpha, t)|^p d\beta(t), \quad (4.7.7)$$

where the measure $d\beta$ is or is not related to $d\alpha$.

These are the subjects I want to discuss here. As a warm-up exercise I prove the following result, which was first published in [Ne19, Theorem 7.31, p. 138]:

THEOREM 4.7.1 [Ne19]. *Let $d\alpha$ be supported in $[-1, 1]$ and let $p \geq 2$. Then for all nonnegative $d\alpha$ -measurable functions w , we have*

$$\begin{aligned} \pi^{-p/2} \int_{\mathbb{R}} |\alpha'(t)(1-t^2)^{1/2}|^{-p/2} w(t) d\alpha(t) \\ \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}} |p_n(d\alpha, t)|^p w(t) d\alpha(t). \end{aligned} \quad (4.7.8)$$

Proof of Theorem 4.7.1. By the triangle inequality,

$$\begin{aligned} \left[\int_{\mathbb{R}} |n^{-1} \lambda_n(d\alpha, t)^{-1}|^{p/2} w(t) d\alpha(t) \right]^{2/p} \\ \leq n^{-1} \sum_{k=0}^{n-1} \left[\int_{\mathbb{R}} |p_k(d\alpha, t)|^p w(t) d\alpha(t) \right]^{2/p}. \end{aligned} \quad (4.7.9)$$

The extremal property (4.1.1) satisfied by the Christoffel functions and Theorem 4.5.1 imply

$$\pi^{-1} \alpha'(t)^{-1} (1-t^2)^{-1/2} \leq \liminf_{n \rightarrow \infty} n^{-1} \lambda_n(d\alpha, t)^{-1} \quad (4.7.10)$$

for almost every t in $[-1, 1]$ whenever $\text{supp}(d\alpha) \subset [-1, 1]$. Thus, by Fatou's lemma, the theorem follows from inequalities (4.7.9) and (4.7.10). ■

The usefulness of Theorem 4.7.1 lies in the possibility of concluding that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |p_n(d\alpha, t)|^p w(t) d\alpha(t) = 0 \quad (4.7.11)$$

implies $w=0$ almost everywhere, under fairly mild conditions on the measure, p and w .

Now I will elaborate on the quadrature sum estimate (4.7.3). There used to be two ways of approaching the problem: the first was introduced by Askey [As4, As5], while the second was developed in [Ne16, Ne17, Ne19, Ne26, Ne30]. If I had seven wishes to be met by a genie, one of them would request the possibility of representing every polynomial Π by means of an integral operator

$$\Pi(x) = \int_{\mathbb{R}} \Pi(t) Q_n(d\alpha, x, t) d\alpha(t), \quad (4.7.12)$$

where Q_n is a nonnegative polynomial in x of degree at most $2n - 1$. If we had (4.7.12), then by Jensen's inequality [PolySz, Vol 1, Sect. 2.1, Problem 71],

$$|\Pi(x)|^p \leq \int_{\mathbb{R}} |\Pi(t)|^p Q_n(d\alpha, x, t) d\alpha(t) \quad (4.7.13)$$

would follow for all $p \geq 1$, and applying the Gauss–Jacobi quadrature formula (3.4) would immediately be obtained with $K = 1$. It is too bad that such genies do not exist, or do they? Moreover, it is evident that it is not reasonable to expect representations of the form (4.7.12) without some additional restrictions. For instance, the degree of the polynomial Π may not be arbitrary. At this point, the classical analyst hiding in us will say “Ho, ho, ho!” There are a number of positive operators with polynomial kernels; for instance, the arithmetic (Fejér) means σ_n of trigonometric Fourier series do such a job in the trigonometric case. Then one should be able to form the delayed (de la Vallée–Poussin) means V_n , that is,

$$V_n = 2\sigma_{2n} - \sigma_n. \quad (4.7.14)$$

These means are trigonometric polynomials of degree at most $2n - 1$, and they leave n th-degree trigonometric polynomials invariant. Moreover, and this is the meat of the matter, the kernel of V_n is the difference of two positive kernels of degree at most $2n - 1$. Thus the feasibility of the representation of all n th-degree polynomials Π in the form

$$\Pi(x) = \int_{\mathbb{R}} \Pi(t) [Q_n^*(d\alpha, x, t) - Q_n^{**}(d\alpha, x, t)] d\alpha(t), \quad (4.7.15)$$

where Q_n^* and Q_n^{**} are nonnegative polynomials of degree at most $2n - 1$ satisfying

$$\sup_{n \geq 1} \int_{\mathbb{R}} [Q_n^*(d\alpha, x, t) + Q_n^{**}(d\alpha, x, t)] d\alpha(t) < \infty, \quad (4.7.16)$$

is no longer that remote. Naturally, if we have (4.7.15) and (4.7.16), we can apply Jensen's inequality and the Gauss–Jacobi quadrature formula to (4.7.15) and then, by (4.7.16), we can estimate the quadrature sums involving the kernel functions.

This is exactly how Askey [As4, As5] reasoned while establishing the first estimate of quadrature sums in terms of integrals.

THEOREM 4.7.2 [As5]. *Let $d\alpha = d\alpha^{(a,b)}$ be a Jacobi distribution in $[-1, 1]$ with parameters a and b , and let $p \geq 1$. If $a \geq b$ and either (i) $a \geq -\frac{1}{2}$ and $b \geq -\frac{1}{2}$ or (ii) $|a - j| \leq 1 + b$ and $-1 < b < -\frac{1}{2}$ for some j such that $2j = 2, 3, \dots$, then*

$$\sum_{k=1}^n |\Pi(x_{kn}(d\alpha))|^p \lambda_{kn}(d\alpha) \leq K \int_{\mathbb{R}} |\Pi(x)|^p d\alpha(x) \quad (4.7.17)$$

for all polynomials Π of degree at most $n - 1$, where $K = K(a, b, p)$.

It was this result of Askey which brought me to the problems discussed here. My first goal was to extend (4.7.17) to all $a > -1$ and $b > -1$. I did not take long for me to realize that I lacked the necessary knowledge to go along the path paved by Askey, which includes positivity results for connection coefficients for hypergeometric functions, a subject I knew nothing about in 1974. Hence I had two options to choose from: either I give up the hope of proving anything of any value about quadrature sums or I take a short cut. Well, retrospectively, I am happy that I chose the latter, especially since, as it turned out later, my approach to proving (4.7.3) for Jacobi polynomials actually yielded a general technique applicable in a variety of situations including generalized Jacobi polynomials, Hermite polynomials, Laguerre polynomials and any other case where one has at one's disposal a suitable Markov–Bernstein inequality, that is, an inequality relating one norm of the derivative of a polynomial to another norm of the polynomial itself.

My method of proving inequalities of type (4.7.3) is based on Christoffel functions, Markov–Stieltjes and Markov–Bernstein inequalities and estimates of consecutive zeros of orthogonal polynomials. As an illustration, I will show how my method works in the example of Chebyshev weights where it is easiest to convey ideas, and then I will formulate some of the general results obtained this way. This approach works only for $p \geq 1$.

Let dT denote the Chebyshev distribution, that is, $dT = v dt$, where

$$v(t) = (1 - t^2)^{-1/2} (|t| < 1) \quad \text{and} \quad v(t) = 0 (|t| \geq 1). \quad (4.7.18)$$

Then $\lambda_{kn}(dT) = \pi/n$ (cf. (4.1.5)), and thus (4.7.3) can be expressed as

THEOREM 4.7.3 [Ne16]. *Let $p \geq 1$. Then, for $\Pi \in \mathbb{P}_m$, the inequality*

$$\sum_{k=1}^n |\Pi(x_{kn})|^p \leq Kn \int_{\mathbb{R}} |\Pi(t)|^p v(t) dt, \quad x_{kn} = \cos((2k+1)\pi/(2n)), \quad (4.7.19)$$

holds where $K = 2\pi^{-1} + m(3p+1)n^{-1}$.

Proof of Theorem 4.7.3

We break the proof of (4.7.19) into two steps.

Step 1. We show that, for all $p > 0$ and all polynomials Π of degree at most m ,

$$\max_{|x| \leq 1} |\Pi(x)|^p \leq m(p+1) 2^{-1} \int_{\mathbb{R}} |\Pi(t)|^p v(t) dt \quad (4.7.20)$$

Proof. Indeed, by (4.1.5),

$$\lambda_n(dT, x)^{-1} \leq (2n-1)\pi^{-1} \leq n, \quad (4.7.21)$$

and thus, by the extremal property (4.1.1),

$$\max_{|x| \leq 1} |R_N(x)|^2 \leq N \int_{\mathbb{R}} |R_N(t)|^2 v(t) dt \quad (4.7.22)$$

for every polynomial R_N of degree at most N . Let d denote the least even integer $\geq p$. Then $\Pi^{d/2}$ is a polynomial of degree $md/2 \leq m(p+1)/2$, and hence, by (4.7.22),

$$\max_{|x| \leq 1} |\Pi(x)|^d \leq m(p+1) 2^{-1} \int_{\mathbb{R}} |\Pi(t)|^d v(t) dt. \quad (4.7.23)$$

Consequently

$$\begin{aligned} \max_{|x| \leq 1} |\Pi(x)|^d &\leq m(p+1) 2^{-1} \int_{\mathbb{R}} |\Pi(t)|^{p+(d-p)} v(t) dt \\ &\leq m(p+1) 2^{-1} \int_{\mathbb{R}} |\Pi(t)|^p v(t) dt \max_{|t| \leq 1} |\Pi(t)|^{d-p}, \end{aligned} \quad (4.7.24)$$

and now (4.7.20) follows directly. ■

Step 2. We show that for all $p \geq 1$ and for all polynomials Π of degree at most m ,

$$\sum_{k=2}^{n-1} |\Pi(x_{kn})|^p \leq (2n\pi^{-1} + 2mp) \int_{-1}^1 |\Pi(t)|^p v(t) dt. \quad (4.7.25)$$

Proof. We start with observing

$$|\Pi(x_{kn})|^p \leq |\Pi(t)|^p + p \int_{x_{k+1,n}}^{x_{k-1,n}} |\Pi(t)|^{p-1} |\Pi'(t)| v(t) dt \quad (4.7.26)$$

for $x_{k+1,n} \leq t \leq x_{k-1,n}$. Next, we use the Markov–Stieltjes inequalities according to which

$$\lambda_{kn}(d\alpha) \leq \int_{x_{k+1,n}}^{x_{k-1,n}} d\alpha(t), \quad k = 2, 3, \dots, n-1, \quad (4.7.27)$$

for all measures $d\alpha$ [Fr31b, Sect. 1.5, pp. 26–33]. By (4.7.26), (4.7.27) and because $\lambda_{kn}(dT) = \pi/n$ (cf. (4.1.5)), we have

$$\sum_{k=2}^{n-1} |\Pi(x_{kn})|^p \leq 2n\pi^{-1} \int_{-1}^1 |\Pi(t)|^p v(t) dt + p \sum_{k=2}^{n-1} \int_{x_{k+1,n}}^{x_{k-1,n}} |\Pi(t)|^{p-1} |\Pi'(t)| dt. \quad (4.7.28)$$

Hence

$$\sum_{k=2}^{n-1} |\Pi(x_{kn})|^p \leq 2n\pi^{-1} \int_{-1}^1 |\Pi(t)|^p v(t) dt + 2p \int_{-1}^1 |\Pi(t)|^{p-1} |\Pi'(t)| dt. \quad (4.7.29)$$

By Hölder's inequality

$$\begin{aligned} & \int_{-1}^1 |\Pi(t)|^{p-1} |\Pi'(t)| dt \\ & \leq \left[\int_{-1}^1 |\Pi(t)|^p v(t) dt \right]^{(p-1)/p} \left[\int_{-1}^1 |\Pi'(t)/v(t)|^p v(t) dt \right]^{1/p}. \end{aligned} \quad (4.7.30)$$

The second integral on the right-hand side of (4.7.30) can be estimated by Bernstein's inequality in L_p spaces [Zy2, p. 11], and we obtain

$$\int_{-1}^1 |\Pi(t)|^{p-1} |\Pi'(t)| dt \leq m \int_{-1}^1 |\Pi(t)|^p v(t) dt. \quad (4.7.31)$$

Now (4.7.25) follows from inequalities (4.7.29) and (4.7.31). ■

Now let us analyze the proof of Theorem 4.7.3. In Step 1 we essentially established that

$$|\Pi(x_{kn})|^p \lambda_{kn} \leq K \int_{\mathbb{R}} |\Pi(x)|^p d\alpha(x) \quad (4.7.32)$$

for $k = 1, 2, \dots, n$; that is, at least for the individual terms on the left-hand side of (4.7.3) we have the right inequality. This can be done for a significantly wider class of measures via generalized Christoffel functions $\lambda_n(d\alpha, p)$ defined in [Ne19, Chap. 6.3] as

$$\lambda_n(d\alpha, p, x) = \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(x) = 1}} \int_{\mathbb{R}} |\Pi(t)|^p d\alpha(t); \quad (4.7.33)$$

they were studied there extensively for generalized Jacobi weights, which we introduce shortly. In Step 2 the essential ingredients were the Markov–Stieltjes inequality [Fr31b, Sect. 1.5], accurate asymptotics for the Christoffel functions and the distances between consecutive zeros of the orthogonal polynomials, and Markov–Bernstein inequalities of the type

$$\int_{-1}^1 |\Pi'(t)/v(t)|^p d\alpha(t) \leq Km^p \int_{-1}^1 |\Pi(t)|^p d\alpha(t) \quad (4.7.34)$$

(cf. (4.7.18) for v) valid for all polynomials $\Pi \in \mathbb{P}_m$. Again, all this has been worked out in [Ne19] and relevant papers such as [Ne16, Ne17, Ne21, Ne26, Ne30, and MăNe1]. Inserting all this information into the skeleton provided by the proof of Theorem 4.7.3, we obtain

THEOREM 4.7.4 [Ne19]. *Let $d\alpha$ be a generalized Jacobi distribution, and let $p \geq 1$. Then*

$$\sum_{k=1}^n |\Pi(x_{kn})|^p \lambda_{kn} \leq K(mn^{-1} + 1) \int_{\mathbb{R}} |\Pi(x)|^p d\alpha(x) \quad (4.7.35)$$

for every $\Pi \in \mathbb{P}_m$, where $K = K(d\alpha, p)$.

Here the measure $d\alpha$ is called a generalized Jacobi distribution if $\text{supp}(d\alpha) = [-1, 1]$ and $d\alpha(t) = w(t) dt$, where

$$w(t) = g(t)(1-t)^{\Gamma_0} \prod_{k=1}^M |t_k - t|^{\Gamma_k} (1+t)^{\Gamma_{M+1}}, \quad -1 \leq t \leq 1, \quad (4.7.36)$$

$-1 < t_M < t_{M-1} < \dots < t_1 < 1$, $\Gamma_k > -1$, $k = 0, 1, \dots, M+1$, and $g^{\pm 1} \in L_\infty$ in $[-1, 1]$.

At this point the reader must have observed that neither Askey's nor my method enables one to extend (4.7.3) to $0 < p < 1$. In the first case, the reason for this is that Jensen's inequality works only with convex functions, whereas in the second case $p-1$ becomes negative when $p < 1$, and thus Hölder's inequality cannot be applied in (4.7.30). The extension to $0 < p < 1$

was made in the recent paper [LuMáNe], where our Hegelian dialectics led us back to (4.7.12) and then we took the courageous leap from the impossible (4.7.12) to the very much possible (4.7.13) or, more accurately, to

$$|\Pi(x)|^p \leq K \int_{\mathbb{R}} |\Pi(t)|^p Q_n(d\alpha, x, t) d\alpha(t), \quad (4.7.37)$$

where $K = K(d\alpha, p)$. As before, for methodological reasons I limit myself to discussing the case of the Chebyshev distribution dT (cf. (4.7.18)).

THEOREM 4.7.5 [LuMáNe]. *Let $0 < p < \infty$. Then the inequality*

$$\begin{aligned} |\Pi(x)|^p &\leq K(1 + mn^{-1}) \lambda_n(dT, x) \int_{\mathbb{R}} |\Pi(t)|^p K_n(dT, x, t)^2 dT(t) \\ &= K(1 + mn^{-1}) G_n(dT, |\Pi|^p, x), \quad -1 \leq x \leq 1 \end{aligned} \quad (4.7.38)$$

(cf. (4.5.6)), holds for every polynomial Π of degree at most m , where K depends on p only.

Proof of Theorem 4.7.5. For a given $p > 0$, let us choose an integer L such that $Lp > 2$. Then $\Pi K_n(dT, x, \cdot)^L$ is a polynomial of degree at most $m + nL$, and thus, by (4.7.20),

$$\max_{|x| \leq 1} |\Pi(x) K_n(dT, x, x)^L|^p \leq K(m + Ln) \int_{\mathbb{R}} |\Pi(t) K_n(dT, x, t)^L|^p dT(t). \quad (4.7.39)$$

We have $K_n(dT, x, t) \leq 2n/\pi$ and $K_n(dT, x, x) \geq n/(2\pi)$ for all x and t in $[-1, 1]$ (cf. (4.1.5) and [Fr31b, p. 104]). Since $Lp \geq 2$, we obtain

$$\begin{aligned} n^{Lp} \max_{|x| \leq 1} |\Pi(x)|^p &\leq K(m + Ln) \int_{\mathbb{R}} |\Pi(t)|^p |K_n(dT, x, t)|^{2 + (Lp-2)} dT(t) \\ &\leq Kn^{Lp-2}(m + Ln) \int_{\mathbb{R}} |\Pi(t)|^p K_n(dT, x, t)^2 dT(t), \end{aligned} \quad (4.7.40)$$

that is,

$$\max_{|x| \leq 1} |\Pi(x)|^p \leq Kn^{-1}(mn^{-1} + L) \int_{\mathbb{R}} |\Pi(t)|^p K_n(dT, x, t)^2 dT(t), \quad (4.7.41)$$

and thus, as $\lambda_n(dT, x) \geq \pi/(2n)$ [Fr31b, Theorem 3.3.4, p. 105], the theorem follows. ■

Theorem 4.7.5 can easily be extended to generalized Jacobi distributions defined by (4.7.36). For w given by (4.7.36), let us define w_n by

$$w_n(t) = g(t)[(1-t)^{1/2} + 1/n]^{(2J_0)+1} \\ \times \prod_{k=1}^M [|t_k - t| + 1/n]^{r_k} [(1+t)^{1/2} + 1/n]^{(2J_{M+1})+1}, \quad -1 \leq t \leq 1. \quad (4.7.42)$$

On the basis of my results regarding Christoffel functions of generalized Jacobi distributions [Ne19, Chap. 6.3], we can prove

THEOREM 4.7.6 [LuMáNe]. *Let $d\alpha$ be a generalized Jacobi distribution in the sense of (4.7.36), $0 < p < \infty$, $L \geq 0$, and let l be a positive integer. Assume ψ is an increasing, convex and nonnegative function on the positive real line. Then for all polynomials Π of degree at most ln ,*

$$\psi(|\Pi(x)|^p) w_n(x) \leq C_1 n^{-L+1} \int_{\mathbb{R}} \psi(C_2 |\Pi(t)|^p) |K_n(dT, x, t)|^L d\alpha(t). \quad (4.7.43)$$

Here C_1 and C_2 are constants independent of n , x and Π .

Theorem 4.7.6 not only enables one to prove (4.7.3) for generalized Jacobi distributions, but also makes it possible to relate quadrature sums to the Large Sieve of number theory (cf. [Mo, p. 548, and Theorem 3, p. 559]). As a matter of fact, (4.7.43) provides a convenient means to extend the Large Sieve to algebraic polynomials in weighted L_p spaces.

Recall that the Large Sieve is an inequality for trigonometric polynomials S_n of degree at most n which states that

$$\sum_{k=1}^m |S_n(t_k)|^2 \leq \{2n + \delta^{-1}\} \int_0^{2\pi} |S_n(t)|^2 dt, \quad (4.7.44)$$

whenever $0 \leq t_1 < t_2 < \dots < t_m \leq 2\pi$ and $\delta = \min\{t_2 - t_1, t_3 - t_2, \dots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\} > 0$.

On the basis of the Large Sieve and Theorem 4.7.6, D. S. Lubinsky, A. Máté, and I succeeded in applying purely L_2 techniques to prove.

THEOREM 4.7.7 [LuMáNe]. *Let $d\alpha$ be a generalized Jacobi distribution in the sense of (4.7.36), $0 < p < \infty$, and let l be a positive integer. Assume ψ is an increasing, convex and nonnegative function on the positive real line. Given*

$$-1 \leq y_m < y_{m-1} < \dots < y_1 \leq 1, \quad (4.7.45)$$

set $\theta_j = \arccos y_j \in [0, \pi]$, $j = 1, 2, \dots, m$, and let $\delta = \min\{\theta_2 - \theta_1, \theta_3 - \theta_2, \dots, \theta_m - \theta_{m-1}\} > 0$. Then, for all polynomials Π of degree at most ln ,

$$\sum_{j=1}^m \psi(|\Pi(y_j)|^p w_n(y_j)) \leq C_1 \{n + \delta^{-1}\} \int_{\mathbb{R}} \psi(C_2 |\Pi(t)|^p) d\alpha(t) \quad (4.7.46)$$

and

$$\sum_{j=1}^m \psi(|\Pi(y_j)|^p) \lambda_n(d\alpha, y_j) \leq C_1 \{1 + (n\delta)^{-1}\} \int_{\mathbb{R}} \psi(C_2 |\Pi(t)|^p) d\alpha(t). \quad (4.7.47)$$

Here C_1 and C_2 are constants independent of m , n , δ , Π and $\{y_j\}$, $j = 1, 2, \dots, m$. In particular, if $m = n$ and $y_j = x_{jn}(d\alpha)$, then $(n\delta)^{-1}$ is uniformly bounded, and thus (4.7.47) takes the form

$$\sum_{j=1}^n \psi(|\Pi(x_{jn})|^p) \lambda_{jn} \leq C_1 \int_{\mathbb{R}} \psi(C_2 |\Pi(t)|^p) d\alpha(t). \quad (4.7.48)$$

I will return to (4.7.3) for measures with unbounded support in Section 4.19.

Now I proceed to discuss lower bounds for (4.7.5) and (4.7.6). Such estimates were thoroughly investigated in [Ne19, Chap. 9]. Naturally, these problems are difficult only when we do not have lower bounds for the individual terms $|p_{n-1}(d\alpha, x_{kn})|^p \lambda_{kn}$. For instance, for generalized Jacobi distributions, I proved the following result in [Ne19, Theorem 6.3.28, p. 120, and Theorem 9.31, p. 170].

THEOREM 4.7.8 [Ne19]. *Let $d\alpha$ be a generalized Jacobi distribution in the sense of (4.7.36). Then*

$$C_1 \leq n\lambda_n(d\alpha, x)/w_n(x) \leq C_2, \quad -1 \leq x \leq 1, \quad (4.7.49)$$

for $n = 1, 2, \dots$, where the positive constants C_1 and C_2 do not depend on x and n (cf. (4.7.42)). If, in addition, the modulus of continuity ω of g in (4.7.36) satisfies $\omega(t)/t \in L_1$ in $[0, 1]$, then

$$C_3 \leq w(x_{kn})(1 - x_{kn}^2)^{-1/2} p_{n-1}(d\alpha, x_{kn})^2 \leq C_4, \quad k = 1, 2, \dots, n, \quad (4.7.50)$$

for $n = 1, 2, \dots$, where C_3 and C_4 are positive constants independent of k and n .

Theorem 4.7.8 immediately implies

THEOREM 4.7.9 [Nevai]. *Let $d\alpha$ be a generalized Jacobi distribution in the sense of (4.7.36) and assume that the modulus of continuity ω of g in*

(4.7.36) satisfies $\omega(t)/t \in L_1$ in $[0, 1]$. Let Δ be a fixed subinterval of $[-1, 1]$. Then

$$\liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Delta} |p_{n-1}(d\alpha, x_{kn})|^p \lambda_{kn} > 0 \quad (4.7.51)$$

for every $p > 0$.

The example of Hermite polynomials shows that, in general, neither (4.7.5) nor (4.7.6) need be bounded away from 0. Moreover, for general measures, the problem is so much more difficult that, at the present time, it has been only partially resolved. I proved the following result in [Ne19, Lemma 9.9, p. 159].

THEOREM 4.7.10 [Ne19]. *Let $d\alpha$ be supported in $[-1, 1]$, and assume that $\log \alpha'(\cos t)$ is integrable in $[0, \pi]$. Then*

$$\liminf_{n \rightarrow \infty} \sum_{k=1}^n |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} \geq \pi^{1/2} D(\rho, 0)/2, \quad (4.7.52)$$

where $\rho(t) = \alpha'(\cos t)$ (cf. (3.25) for the definition of Szegő's function D).

Proof of Theorem 4.7.10. Let $n \geq 1$. By the Gauss–Jacobi quadrature formula (3.4), we have

$$2^{n-2} = \gamma_{n-1}(d\alpha) \sum_{k=1}^n T_{n-1}(x_{kn}) p_{n-1}(d\alpha, x_{kn}) \lambda_{kn}, \quad (4.7.53)$$

where T_{n-1} denotes the Chebyshev polynomial of degree $n-1$ whose leading coefficient is 2^{n-2} and $\gamma_{n-1}(d\alpha)$ denotes the leading coefficient of the orthonormal polynomial $p_{n-1}(d\alpha)$ (cf. (3.1)). By the real line variant of Szegő's Theorem 4.11.1 (cf. [Sz2, Theorem 12.7.1, p. 309]),

$$\lim_{n \rightarrow \infty} \gamma_{n-1}(d\alpha) 2^{2-n} = \pi^{1/2} D(\rho, 0)/2 \quad (4.7.54)$$

and thus the theorem follows since $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$. ■

Regarding (4.7.6), which is crucial for solving Turán's problem on divergence of Lagrange interpolation $L_n(d\alpha, f)$ in spaces $L_p(d\alpha)$ for $p > 2$, I can only prove the following

THEOREM 4.7.11 [Ne19]. *Let $d\alpha$ be supported in $[-1, 1]$, and assume that $\log \alpha'(\cos t)$ is integrable in $[0, \pi]$. Then there exists a number $\delta = \delta(d\alpha) > 0$ such that, if $\Omega \subset [-1, 1]$ is a union of a finite set of disjoint intervals with total length $|\Omega| \geq 2 - \delta$, then*

$$\liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Omega} |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} > 0. \quad (4.7.55)$$

Proof of Theorem 4.7.11. Let $c\Omega = [-1, 1] \setminus \Omega$, and let $1_{c\Omega}$ be the characteristic function of $c\Omega$. We have

$$\begin{aligned} \sum_{k=1}^n |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} &= \sum_{x_{kn} \in \Omega} |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} \\ &\quad + \sum_{k=1}^n 1_{c\Omega}(x_{kn}) |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} \end{aligned} \quad (4.7.56)$$

and by Schwarz' inequality and the Gauss–Jacobi quadrature formula (3.4),

$$\begin{aligned} \sum_{k=1}^n |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} &\leq \sum_{x_{kn} \in \Omega} |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} \\ &\quad + \left[\int_{-1}^1 d\alpha \sum_{k=1}^n 1_{c\Omega}(x_{kn}) p_{n-1}(d\alpha, x_{kn})^2 \lambda_{kn} \right]^{1/2} \end{aligned} \quad (4.7.57)$$

The function $1_{c\Omega}$ is Riemann integrable in $[-1, 1]$, and thus, by Theorem 3.2.3 in [Ne19, p. 17],

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n 1_{c\Omega}(x_{kn}) p_{n-1}(d\alpha, x_{kn})^2 \lambda_{kn} = 2\pi^{-1} \int_{c\Omega} (1-t^2)^{1/2} dt. \quad (4.7.58)$$

Applying Theorem 4.7.10, (4.7.57), and (4.7.58), we obtain

$$\begin{aligned} \pi^{1/2} D(\rho, 0)/2 &\leq \liminf_{n \rightarrow \infty} \sum_{x_{kn} \in \Omega} |p_{n-1}(d\alpha, x_{kn})| \lambda_{kn} \\ &\quad + \left[\int_{-1}^1 d\alpha 2\pi^{-1} \int_{c\Omega} (1-t^2)^{1/2} dt \right]^{1/2}. \end{aligned} \quad (4.7.59)$$

Thus (4.7.55) holds if the Lebesgue measure of $c\Omega$ is sufficiently small. ■

The last problem I discuss in this section concerns estimating (4.7.7). For the trigonometric system, we have Fejér's [Fe] theorem according to which

$$\lim_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} f(t) |\sin nt|^p dt = (2\pi)^{-1} \int_0^{2\pi} f(t) dt (2\pi)^{-1} \int_0^{2\pi} |\sin t|^p dt \quad (4.7.60)$$

for all $p > 0$ [Zy1, Theorem 2.4.15, p. 49]. For general orthogonal polynomials it would be unrealistic to expect to be able to prove similar

results at the present time since one cannot even handle L_p boundedness of such orthogonal polynomials for $p > 2$. Nevertheless, it turns out that a very useful lower estimate can be given for (4.7.7) for a large class of measures.

I started investigating such problems in [Ne19, Chap. 9], where a number of results were obtained, including Theorem 4.7.1. Recent advances in generalizing and extending Szegő's theory (cf. Sections 4.11 and 4.13), however, have made it possible to surpass all previous results in this direction. The following theorem by A. Máté, V. Totik, and me [MáNeTo6, Theorem 2] is a typical product of our extension of Szegő's theory.

THEOREM 4.7.12 [MáNeTo6]. *Let $\text{supp}(d\alpha) = [-1, 1]$, $\alpha' > 0$ almost everywhere in $[-1, 1]$, and suppose $0 < p \leq \infty$. If g is a Lebesgue-measurable function in $[-1, 1]$, then*

$$\begin{aligned} & \pi^{-1/2} \left[\int_{-1}^1 |g(t) \alpha'(t)^{-1/2} (1-t^2)^{-1/4}|^p dt \right]^{1/p} \\ & \leq 2^{\max\{1/p-1/2, 0\}} \liminf_{n \rightarrow \infty} \left[\int_{-1}^1 |g(t) p_n(d\alpha, t)|^p dt \right]^{1/p} \end{aligned} \quad (4.7.61)$$

In particular, if

$$\liminf_{n \rightarrow \infty} \left[\int_{-1}^1 |g(t) p_n(d\alpha, t)|^p dt \right]^{1/p} = 0 \quad (4.7.62)$$

then $g = 0$ almost everywhere.

I conclude this section by formulating one of the basic ingredients in the proof of the previous theorem which itself is one of the loveliest results we ever proved.

THEOREM 4.7.13 [MáNeTo6]. *Let $\text{supp}(d\alpha) = [-1, 1]$, and let $\alpha' > 0$ almost everywhere in $[-1, 1]$. For a given real number c and a nonnegative integer n , define the set $B_{c,n}(d\alpha)$ by*

$$B_{c,n}(d\alpha) = \{t: p_n(d\alpha, t)^2 \alpha'(t)(1-t^2)^{1/2} \geq c\}. \quad (4.7.63)$$

Then, for every $c > 2/\pi$,

$$\lim_{n \rightarrow \infty} |B_{c,n}(d\alpha)| = 0, \quad (4.7.64)$$

where $|E|$ denotes the Lebesgue measure of the set E . Moreover, (4.7.64) does not necessarily hold for $c < 2/\pi$.

In other words, the orthogonal polynomials are uniformly bounded in measure, and the bound is exactly what one would expect. Naturally, by Rahmanov's theorem [Rah3], pointwise boundedness cannot be guaranteed by solely size conditions imposed on α' ; Steklov's conjecture fails to be true.

4.8. Mean Convergence of Lagrange Interpolation

In this section we are concerned with necessary and/or sufficient conditions for weighted mean convergence of Lagrange interpolation taken at zeros of orthogonal polynomials associated with measures with compact support. Throughout this section we assume that the support of the measure $d\alpha$ is in $[-1, 1]$ and f is a real valued function in the same interval. Recall that, for a given f , the Lagrange interpolating polynomial $L_n(d\alpha, f)$ is defined to be the unique algebraic polynomial of degree at most $n-1$ which satisfies

$$L_n(d\alpha, f, x_{kn}) = f(x_{kn}), \quad k = 1, 2, \dots, n, \quad (4.8.1)$$

and it can be expressed as

$$L_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) l_{kn}(d\alpha, x), \quad (4.8.2)$$

where the fundamental polynomials $l_{kn}(d\alpha)$ are given by

$$l_{kn}(d\alpha, x) = \frac{p_n(d\alpha, x)}{p_n'(d\alpha, x_{kn})(x - x_{kn})}, \quad k = 1, 2, \dots, n \quad (4.8.3)$$

(cf. (3.15) and (3.16)). By the Gauss-Jacobi quadrature formula (3.4), we can easily evaluate the $L_2(d\alpha)$ norm of $L_n(d\alpha, f)$ and we obtain

$$\int_{\mathbb{R}} |L_n(d\alpha, f, t)|^2 d\alpha(t) = \sum_{k=1}^n |L_n(d\alpha, f, x_{kn})|^2 \lambda_{kn}; \quad (4.8.4)$$

thus $L_n(d\alpha, f)$, as an operator from C to $L_2(d\alpha)$, is certainly uniformly bounded in n . This is the simple reason why Erdős and Turán's [ErTu1] well-known $L_2(d\alpha)$ convergence result holds:

THEOREM 4.8.1 [ErTu1]. *Let $\text{supp}(d\alpha) \subset [-1, 1]$. Then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f(t) - L_n(d\alpha, f, t)|^2 d\alpha(t) = 0 \quad (4.8.5)$$

for every function f continuous in $[-1, 1]$.

In connection with (4.8.5), it is natural to ask whether one can obtain conditions (in terms of $p > 0$ and $d\beta$) guaranteeing

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |f(t) - L_n(d\alpha, f, t)|^p d\beta(t) = 0 \quad (4.8.6)$$

for all continuous f . Of course, we know that Erdős and E. Feldheim [ErFe] proved (4.8.6) for all $p > 0$ when both measures $d\alpha$ and $d\beta$ are the Chebyshev distribution dT (cf. (4.7.18)). It is interesting that both Freud and Turán agreed that the resolution of this problem is of primary significance. Freud lists this as an unsolved problem No. 1 in [Fr31b, p. 273] whereas Turán discusses it in [Tu2, p. 186; Tu4, pp. 31–34].

One of Turán's favorite and frequently repeated problems was the following one, last published in [Tu4, Problem VIII, p. 32].

PROBLEM (Turán). Does there exist an absolutely continuous measure $d\alpha$ with support in $[-1, 1]$ such that for some continuous function f , we have

$$\limsup_{n \rightarrow \infty} \int_{-1}^1 |f(t) - L_n(d\alpha, f, t)|^p d\alpha(t) = \infty \quad (4.8.7)$$

for every $p > 2$?

Neither Turán nor Freud knew the answer to this problem. It was Askey [As4] who gave the right answer (yes) and it was I who proved it in [Ne34].

At this point I cannot resist the temptation to tell the following story. When I first discussed my mathematical future with Turán in 1970, he told me that if I ever wanted to prove significant results in approximation theory and orthogonal polynomials, the most important thing was to study Askey's papers, especially the one dealing with mean convergence of Lagrange interpolation [As4]. I consulted Freud, as well, regarding the kind of studies and research I should undertake, and his advice was essentially identical. Freud suggested that I investigate weighted L_p convergence of Lagrange interpolation, and he recommended that I get in touch with Askey, who had the most promising results in this direction. I find it touching that the well-known (somewhat tragic, somewhat comic) feud between Freud and Turán notwithstanding, they had such similar mathematical tastes. I listened to both of them, and this is how I started drifting towards Askey, who in the long run became responsible to some extent for my continued and deeply rooted interest in orthogonal polynomials.

When I recently asked Askey how he came to believe that there are weights such that (4.8.7) holds for every $p > 2$, he told me that when he

proved $L_p(d\alpha)$ convergence of Lagrange interpolation taken at zeros of ultraspherical polynomials (cf. Theorem 4.8.5), he noticed that the exponent p in (4.8.6) for which (4.8.6) is satisfied (with $d\beta = d\alpha$) is such that $p \rightarrow 2$ as the parameter in the ultraspherical weights tends to ∞ . Thus he concluded that if one picks a weight which is flatter than any of the ultraspherical ones, then that weight certainly must satisfy the conditions in Turán's problem. An example of such a weight is given by the Pollaczek weight (cf. (4.13.8)). What is wonderful about this reasoning is that it actually works, though it took another person (me) and another 15 years to prove it rigorously. As it turned out, the solution came step by step via applications of the results discussed in Section 4.7, above. Askey's philosophy is crystallized in the following theorem, which I proved in [Ne34].

THEOREM 4.8.2 [Ne34]. *Let $\text{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos \theta) \in L_1$ in $[0, 2\pi]$. Let $1 \leq p_0 < \infty$ and $u (\geq 0) \in L_1$ in $[-1, 1]$. Suppose that*

$$\int_{-1}^1 [\alpha'(t)(1-t^2)^{1/2}]^{-p/2} u(t) dt = \infty \quad (4.8.8)$$

for every $p > p_0$. Then there exists a continuous function f such that

$$\limsup_{n \rightarrow \infty} \int_{-1}^1 |L_n(d\alpha, f, t)|^p u(t) dt = \infty \quad (4.8.9)$$

for every $p > p_0$.

Although the proof of Theorem 4.8.2 is beyond the scope of this survey, I will nevertheless elaborate on some of the details which are the main building blocks in the proof.

Sketch of the Proof of Theorem 4.8.2

Step 1 [Ne19, Theorem 10.15, p. 180]. We show that if (4.8.8) holds for a single p , then there is a continuous function f such that (4.8.9) is satisfied. Our starting point is the following expression [Fr31b, Formula (3.6.3), p. 114] for the fundamental polynomials in (4.8.3),

$$l_{kn}(d\alpha, x) = a_n(d\alpha) \lambda_{kn}(d\alpha) p_{n-1}(d\alpha, x_{kn}) p_n(d\alpha, x) / (x - x_{kn}) \quad (4.8.10)$$

(cf. (3.5)–(3.8) and (3.13)). By (4.8.10) and Theorem 4.7.11, if \mathcal{A} is a sufficiently small interval, then there is a continuous function f_n such that $|f_n| \leq 1$ and

$$|L_n(d\alpha, f_n, x)| \geq K a_n(d\alpha) |p_n(d\alpha, x)|, \quad x \in \mathcal{A}, \quad (4.8.11)$$

where K is a positive constant independent of Δ and n (just take a suitable sawtooth function). Hence

$$\begin{aligned} & [Ka_n(d\alpha)]^p \int_{\Delta} |p_n(d\alpha, t)|^p u(t) dt \\ & \leq \int_{\Delta} |L_n(d\alpha, f_n, t)|^p u(t) dt \\ & \leq \sup_{\|f\|_C \leq 1} \int_{-1}^1 |L_n(d\alpha, f, t)|^p u(t) dt. \end{aligned} \tag{4.8.12}$$

By Rahmanov's Theorem 4.5.7, the recurrence coefficients $a_n(d\alpha)$ tend to $\frac{1}{2}$ as $n \rightarrow \infty$. Thus by Theorem 4.7.12,

$$\begin{aligned} & K \int_{\Delta} [\alpha'(t)(1-t^2)^{1/2}]^{-p/2} u(t) dt \\ & \leq \liminf_{n \rightarrow \infty} \sup_{\|f\|_C \leq 1} \int_{-1}^1 |L_n(d\alpha, f, t)|^p u(t) dt, \end{aligned} \tag{4.8.13}$$

where $K > 0$ is independent of Δ . By (4.8.8), there is an interval Δ such that

$$\int_{\Delta} [\alpha'(t)(1-t^2)^{1/2}]^{-p/2} u(t) dt = \infty, \tag{4.8.14}$$

and therefore, by (4.8.13),

$$\liminf_{n \rightarrow \infty} \sup_{\|f\|_C \leq 1} \int_{-1}^1 |L_n(d\alpha, f, t)|^p u(t) dt = \infty \tag{4.8.15}$$

Now the existence of a continuous function f such that (4.8.9) is satisfied follows from the uniform boundedness principle. ■

Step 2. The existence of the omnipotent continuous function f in (4.8.9) is guaranteed by the following technical proposition about sequences of operators on families of Banach spaces [Ne34, Lemma].

THEOREM 4.8.3 [Ne34]. *Let D be a Banach space with norm $\|\cdot\|$ and let $\{B_p\}$, $p_0 \leq p \leq \infty$, be a collection of Banach spaces B_p with norm $\|\cdot\|_p$ such that $B_p \subset B_q$ for $p > q$ and $\|b\|_q \leq \|b\|_p$ if $q < p$ and $b \in B_p$. Let $\{L_n\}$, $n = 1, 2, \dots$, be a sequence of bounded linear operators defined on D with values in B_∞ such that*

$$\lim_{n \rightarrow \infty} \sup_{\|f\| \leq 1} \|L_n(f)\|_p = \infty \tag{4.8.16}$$

for every $p_0 < p \leq \infty$. Then there exists an $f \in D$ such that

$$\limsup_{n \rightarrow \infty} \|L_n(f)\|_p = \infty \quad (4.8.17)$$

for every $p_0 < p \leq \infty$. ■

Theorem 4.8.2 goes beyond solution of Turán's problem, which can be obtained from the former by setting $u = \alpha'$. I wish also to point out that one can use Theorem 10.19 in [Ne19, p. 182] to prove a variant of Theorem 4.8.3 for L_p spaces with $0 < p < 1$, and that would extend Theorem 4.8.2 for the case when $0 < p < \infty$. Applying Theorem 10.16 in [Ne19, p. 181], one can produce versions of Theorem 4.8.2 where the condition $\log \alpha'(\cos \theta) \in L_1$ is replaced by other requirements. It is also easy to see that $n \rightarrow \infty$ in (4.8.9) can be weakened to $n_j \rightarrow \infty$, where $\{n_j\}$ is any given increasing sequence of positive integers.

Theorem 4.8.2 is a negative result for a wide class of measures. For wide classes of projection operators, R. Nessel and his group obtained a number of results of very general character (cf. [GöMa2] and the references therein). Now let us turn our attention to positive results regarding mean convergence of Lagrange interpolation. Although it is not true that, in the general case, (4.8.8) is necessary and sufficient for (4.8.9) (cf. [Ne30, Theorem 7, p. 696]), it turns out that if both u and α' are Jacobi weights, or generalized Jacobi weights, then (4.8.8) and (4.8.9) are indeed equivalent. The first nontrivial results in this direction were discovered by Askey [As4, As5], who revived an old idea of J. Marcinkiewicz [Mar] which succeeds in reducing the proof of L_p convergence of Lagrange interpolation to that of orthogonal Fourier series and, what is even more amazing, accomplishes this via L_2 arguments. In what follows I briefly elaborate on Askey's method.

Let \mathbb{A} denote the class of measures $d\alpha$ for which there is a constant $K > 0$ such that

$$\sum_{k=1}^n |\Pi(x_{kn})| \lambda_{kn} \leq K \int_{\mathbb{R}} |\Pi(x)| d\alpha(x) \quad (4.8.18)$$

whenever Π is a polynomial of degree less than n (cf. (4.7.3)).

I summarize Askey's method in [As4] as

THEOREM 4.8.4. *Let $d\alpha \in \mathbb{A}$ and let $d\beta$ be absolutely continuous with respect to $d\alpha$. Then, for every $1 \leq p < \infty$, we have*

$$\sup_{\|f\|_C \leq 1} \int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) \leq K^p \sup_{\|f\|_C \leq 1} \int_{\mathbb{R}} |S_n(d\alpha, f, t)|^p d\beta(t), \quad (4.8.19)$$

where $S_n(d\alpha, f)$ denotes the partial sum of the orthogonal Fourier series of the function f (cf. (3.10)) and K is the constant in (4.8.18).

Proof of Theorem 4.8.4. Since $d\beta$ is absolutely continuous with respect to $d\alpha$, we can write $d\beta = g d\alpha$. Then

$$\begin{aligned} & \int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) \\ &= \int_{\mathbb{R}} L_n(d\alpha, f, t) [\text{sign } L_n(d\alpha, f, t)] |L_n(d\alpha, f, t)|^{p-1} g(t) d\alpha(t) \\ &= \int_{\mathbb{R}} L_n(d\alpha, f, t) G(t) d\alpha(t), \end{aligned} \quad (4.8.20)$$

where

$$G(t) = [\text{sign } L_n(d\alpha, f, t)] |L_n(d\alpha, f, t)|^{p-1} g(t). \quad (4.8.21)$$

If we expand G in the orthogonal Fourier series $S(d\alpha, G)$, then the partial sums $S_n(d\alpha, G)$ satisfy

$$\int_{\mathbb{R}} L_n(d\alpha, f, t) G(t) d\alpha(t) = \int_{\mathbb{R}} L_n(d\alpha, f, t) S_n(d\alpha, G, t) d\alpha(t) \quad (4.8.22)$$

since $P_n(d\alpha)$ is orthogonal to all polynomials of lower degree, and the degree of $L_n(d\alpha, f)$ is at most $n-1$. Hence, by (4.8.20),

$$\int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) = \int_{\mathbb{R}} L_n(d\alpha, f, t) S_n(d\alpha, G, t) d\alpha(t). \quad (4.8.23)$$

The next step is to apply the Gauss–Jacobi quadrature formula (3.4) to the right side of (4.8.23). Taking the interpolating property of $L_n(d\alpha, f)$ into consideration, we obtain

$$\begin{aligned} \int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) &= \sum_{k=1}^n L_n(d\alpha, f, x_{kn}) S_n(d\alpha, G, x_{kn}) \lambda_{kn} \\ &= \sum_{k=1}^n f(x_{kn}) S_n(d\alpha, G, x_{kn}) \lambda_{kn} \\ &\leq \|f\|_C \sum_{k=1}^n |S_n(d\alpha, G, x_{kn})| \lambda_{kn} \end{aligned} \quad (4.8.24)$$

At this point we use (4.8.18). Since $d\alpha \in \mathbb{A}$, we have

$$\int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) \leq K \|f\|_C \int_{\mathbb{R}} |S_n(d\alpha, G, t)| d\alpha(t). \quad (4.8.25)$$

Let H be defined by

$$H(t) = \text{sign } S_n(d\alpha, G, t). \quad (4.8.26)$$

Then we can repeat our previously applied arguments to conclude

$$\begin{aligned} \int_{\mathbb{R}} |S_n(d\alpha, G, t)| d\alpha(t) &= \int_{\mathbb{R}} S_n(d\alpha, G, t) H(t) d\alpha(t) \\ &= \int_{\mathbb{R}} G(t) S_n(d\alpha, H, t) d\alpha(t). \end{aligned} \quad (4.8.27)$$

Let $q = p/(p-1)$. Then by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}} |S_n(d\alpha, G, t)| d\alpha(t) \\ \leq \left[\int_{\mathbb{R}} |G(t)/g(t)|^q g(t) d\alpha(t) \right]^{1/q} \left[\int_{\mathbb{R}} |S_n(d\alpha, H, t)|^p g(t) d\alpha(t) \right]^{1/p} \end{aligned} \quad (4.8.28)$$

and in view of (4.8.21), (4.8.25), and $d\beta = g d\alpha$, we obtain

$$\int_{\mathbb{R}} |L_n(d\alpha, f, t)|^p d\beta(t) \leq [K \|f\|_C]^p \int_{\mathbb{R}} |S_n(d\alpha, H, t)|^p d\beta(t). \quad (4.8.29)$$

Finally, we observe that H in (4.8.26) is piecewise continuous. Thus (4.8.19) follows from (4.8.29). ■

Theorem 4.7.7 tells us that generalized Jacobi distributions are in the class \mathbb{A} , and this is exactly one of the main reasons why we were interested in estimates of quadrature sums of the form (4.7.3) in Section 4.7. The message conveyed by Theorem 4.8.4 is that, for the class \mathbb{A} , weighted mean convergence of Lagrange interpolation follows from that of orthogonal Fourier series. There is a fairly extensive literature dealing with the latter problem (cf. [AsWa1, Ba1, Ba2, Mu1–Mu3, Pol11–Pol13, Win]). For instance, one can use Theorems 4.7.7 and 4.8.4 and V. Badkov's results in [Ba1] to prove the following convergence theorem for Lagrange interpolation at zeros of smooth generalized Jacobi distributions.

In this section the measure $d\alpha$ is called a smooth generalized Jacobi distribution if $\text{supp}(d\alpha) = [-1, 1]$ and $d\alpha(t) = w(t) dt$, where

$$w(t) = g(t)(1-t)^{\Gamma_0} \prod_{k=1}^m |t_k - t|^{\Gamma_k} (1+t)^{\Gamma_{m+1}}, \quad -1 \leq t \leq 1, \quad (4.8.30)$$

with $-1 < t_m < t_{m-1} < \dots < t_1 < 1$, $\Gamma_k > -1$, $k = 0, 1, \dots, m+1$. Here g

satisfies $g^{\pm 1} \in L_{\infty}$ in $[-1, 1]$ and $\omega(t)/t \in L_1$, where ω is the modulus of continuity of g . We will also say that u is a generalized Jacobi weight if u can be written in the form of the right-hand side of (4.8.30) with $g \equiv 1$.

THEOREM 4.8.5 [Ne30]. *Let $d\alpha$ be a generalized smooth Jacobi distribution and let u be a generalized Jacobi weight. Let $0 < p < \infty$. Then*

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |f(t) - L_n(d\alpha, f, t)|^p u(t) dt = 0 \quad (4.8.31)$$

for every function f continuous on $[-1, 1]$ if and only if

$$\int_{-1}^1 [\alpha'(t)(1-t^2)^{1/2}]^{-p/2} u(t) dt < \infty. \quad (4.8.32)$$

Theorem 4.8.5 generalizes all results previously known on mean convergence of Lagrange interpolation, including those of Erdős and Feldheim [ErFe], Feldheim [Fel1–Fel4], Marcinkiewicz [Mar], and Askey [As4, As5]. To some extent I consider this theorem a tribute to Askey, who in the late sixties, being an unknown approximator (though by then he had already earned a reputation in harmonic analysis), had the courage to enter an area where well-established stars such as Freud and Turán failed to resolve some of their own favorite problems and who came up with a number of partially forgotten and partially fresh ideas which eventually led to a conceptually splendid solution of the basic problems. In all fairness, one should not forget to mention the influence of papers of J. Marcinkiewicz on both Askey's and my research.

One of the limitations of Askey's orthogonal Fourier series method described in Theorem 4.8.4 is that it requires knowledge of convergence of orthogonal Fourier series in the same weighted L_p space where the convergence of Lagrange interpolation is studied. Since at the present time nothing is known on convergence of orthogonal Fourier series in L_p spaces with arbitrary weights (measures), one is forced to search for other approaches when considering convergence of Lagrange interpolation in L_p spaces with general weights and/or measures. In my paper [Ne30], I demonstrated that by realizing that, in fact, Lagrange interpolation can be looked at as a mapping from bounded functions into the appropriate weighted L_p space under consideration rather than as a mapping from L_p into L_p , one can directly estimate and/or evaluate the norms without referring to the relationship between Lagrange interpolation and orthogonal Fourier series as expressed in Theorem 4.8.4. My new method still requires that quadrature sums be handled in a proper way, but the technique described in Section 4.7 is suitable for such a purpose.

In [Ne30] I set the goal of finding necessary and sufficient conditions for convergence of Lagrange interpolation based at zeros of generalized Jacobi polynomials associated with smooth generalized Jacobi distributions in the sense of (4.8.30) in L_p spaces with general weights. As a matter of fact, I considered quasi-Lagrange interpolating polynomials which have the property that they interpolate not just at the zeros of orthogonal polynomials but also possibly at two more exceptional points and, at these exceptional points all their derivatives up to a prescribed order vanish. It turns out that although by doing so we might ruin convergence when ordinary Lagrange interpolation does converge, nevertheless the quasi-Lagrange interpolating polynomials will converge when ordinary Lagrange interpolation does not. This phenomenon is described in the following theorem, which is one of my all-time favorites.

THEOREM 4.8.6 [Ne30]. *Let $d\alpha$ be a generalized smooth Jacobi distribution in the sense of (4.8.30), $0 < p < \infty$, and let r and s be nonnegative integers. Let u be a nonnegative function defined in $[-1, 1]$ such that $u \in (L \log^+ L)_p$ in $[-1, 1]$ and u is positive on a set with positive Lebesgue measure, and let $v(x) = (1-x)^{-r}(1+x)^{-s}$. Let $L_n^{(r,s)}(d\alpha, f)$ be the quasi-Lagrange interpolating polynomial defined by*

$$L_n^{(r,s)}(d\alpha, f, x) = f(x_{kn}), \quad k = 0, 1, 2, \dots, n+1, \quad (4.8.33)$$

where, for $k = 1, 2, \dots, n$, the points x_{kn} are the zeros of the associated orthogonal polynomials, $x_{0n} = 1$, $x_{n+1,n} = -1$ (if either r or s equals 0, then $k = 0$ or $k = n+1$, respectively, is omitted in (4.8.33)),

$$L_n^{(r,s)}(d\alpha, f, 1)^{(l)} = 0, \quad l = 1, 2, \dots, r-1, \quad (4.8.34)$$

and

$$L_n^{(r,s)}(d\alpha, f, -1)^{(l)} = 0, \quad l = 1, 2, \dots, s-1. \quad (4.8.35)$$

Then

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |[f(t) - L_n^{(r,s)}(d\alpha, f, t)] u(t)|^p dt = 0 \quad (4.8.36)$$

for every function f continuous on $[-1, 1]$ if and only if

$$\int_{-1}^1 \alpha'(t)^{1/2} (1-t^2)^{1/4} v(t) dt < \infty \quad (4.8.37)$$

and

$$\int_{-1}^1 [\alpha'(t)^{-1/2} (1-t^2)^{-1/4} u(t)]^p dt < \infty. \quad (4.8.38)$$

Moreover, there exists a nonnegative function u such that $u \in L_p \setminus (L \log^+ L)_p$

in $[-1, 1]$ and conditions (4.8.37)–(4.8.38) do not imply weighted mean convergence in (4.8.36) for every continuous function f .

I wish to point out that both Theorem 4.8.5 and Theorem 4.8.6 concern L_p convergence of interpolation for all $0 < p < \infty$. For $0 < p < 1$, none of the ideas described above are applicable, and the convergence in this case is taken care of via certain delicate inequalities involving integrals with different values of the exponent p . Contrary to one's expectation, Nikolskii-type inequalities cannot be used, and the actual inequalities applied are rather of an ad hoc nature.

I conclude this section with the following quotation from Askey's [As4, p. 84, first paragraph]: "The lack of nice theorems for $p = \frac{4}{3}$ and $p = 4$ (for weighted mean convergence of orthogonal Fourier series in Laguerre polynomials) suggests that there are only fairly weak results to be obtained for Lagrange interpolation at the zeros of the Laguerre or Hermite polynomials. Turán raised this question in [Tur2] and I, too, would like to see some results on this question. However I am afraid that they will be weaker than one might have suspected." I will return to Lagrange interpolation at zeros of Laguerre and Hermite polynomials in Section 4.19. Right now I merely inform the reader that the "weaker than one might have suspected" convergence does actually take place in L_p , for all $p > 1$, with an appropriate weight function. I hope Askey will forgive me for pointing a finger at him. The point is that sometimes even one of the greatest predictors might fall. Why? Well, the reason is that mean convergence of Lagrange interpolation cannot be treated as a purely L_p problem. As a matter of fact, the game has to be played in L_∞ equipped with L_p metric.

4.9. Zeros of Orthogonal Polynomials and Eigenvalues of Toeplitz Matrices

Freud had a number of most interesting papers on zeros of orthogonal polynomials dealing with the case where the corresponding measure is not supported in a finite interval, and I will discuss these in 4.18. Searching through his publication list and my memory, I could find only two papers by Freud treating zeros of orthogonal polynomials associated with measures whose support is compact. One of them is a joint work with Erdős [ErFr], while the other [Fr7] concerns result of Erdős and Turán. The first one is exciting and uses no Christoffel functions, whereas the second is abundant with somewhat routine applications of Christoffel functions. Besides this, he also had some tidbits scattered around in several of his papers on Lagrange and Hermite–Fejér interpolation. Most of these results are duly exposed in his book [Fr31a, b], and thus I am under no pressure and/or obligation to review them here. Instead, I will talk about

results missed by Freud which are of great importance and whose intimate relation to Christoffel functions is much more than just a fleeting adventure.

One of Erdős and Turán's most celebrated results is the one on distribution of zeros of orthogonal polynomials.

THEOREM 4.9.1 [ErTu3]. *Let the measure $d\alpha$ satisfy $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$. Then*

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n f(x_{kn}(d\alpha)) = \pi^{-1} \int_{-1}^1 f(t)(1-t^2)^{-1/2} dt \quad (4.9.1)$$

(cf. (3.5)) for very function f which is Riemann integrable in $[-1, 1]$.

Due to the importance of this theorem, there have been numerous papers treating the limit formula (4.9.1) and its generalizations under various conditions on the measure (cf. [ErFr, Korov1, Korov2, U12, U16, etc.]). As a matter of fact, Szegő's Strong Limit Theorem [GrSz; Sz4, Vol 3, p. 269] regarding Toeplitz determinants is nothing but (4.9.1) with a super-accurate remainder term. What I find incredible is that, for many years, nobody even suspected that (4.9.1) is improvable to a great extent under the sole condition that $\alpha' > 0$ almost everywhere in $[-1, 1]$.

In this section, we will say that the measure $d\alpha$ is in the class \mathbb{M} if the recurrence coefficients $a_n(d\alpha)$ and $b_n(d\alpha)$ in (3.7) satisfy

$$\lim_{n \rightarrow \infty} a_n(d\alpha) = \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n(d\alpha) = 0. \quad (4.9.2)$$

The class \mathbb{M} has been thoroughly studied in [Ne19]. For our purposes it is enough to know that if $d\alpha \in \mathbb{M}$ then $\text{supp}(d\alpha)$ is a compact set containing $[-1, 1]$ and \mathbb{M} is sufficiently large to be of significant interest. More specifically, if $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$, then $d\alpha \in \mathbb{M}$ (cf. Theorem 4.5.7 and the comments thereafter regarding this fundamental result of Rahmanov).

According to the following theorem that I proved in [Ne24, Theorem 9, p. 347], zeros of orthogonal polynomials and Christoffel functions live and thrive together in \mathbb{M} .

THEOREM 4.9.2 [Ne24]. *Let $d\alpha \in \mathbb{M}$. If f is twice continuously differentiable in an interval Δ containing the support of $d\alpha$, then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n f(x_{kn}) - \int_{\mathbb{R}} f(t) \lambda_n(d\alpha, t)^{-1} d\alpha(t) \right] \\ = (2\pi)^{-1} \int_{-1}^1 f(t)(1-t^2)^{-1/2} dt - f(1)/4 - f(-1)/4. \end{aligned} \quad (4.9.3)$$

The proof of (4.9.3) consists of two parts. First, one demonstrates that the expression in the brackets on the left-hand side of (4.9.3) is a bounded functional on the space of twice continuously differentiable functions on Δ with seminorm $\|f\| = \max |f''(t)|$, $t \in \Delta$. This is fairly straightforward. The second step is to verify (4.9.3) for a dense subset, say polynomials. The latter uses all the machinery which was discovered in [Ne19].

Theorem 4.9.2 leads to the following generalization of Erdős and Turán's Theorem 4.9.1, which is another of my all-time favorites [Ne24, Theorem 10, p. 350].

THEOREM 4.9.3 [Ne24]. *Let $d\alpha \in \mathbb{M}$. Then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n f(x_{kn}(d\alpha)) - \sum_{k=1}^{n-1} f(x_{k,n-1}(d\alpha)) \right] \\ = \pi^{-1} \int_{-1}^1 f(t)(1-t^2)^{-1/2} dt \end{aligned} \quad (4.9.4)$$

for every function f which is continuously differentiable in $\Delta \subset \text{supp}(d\alpha)$. In particular, (4.9.4) is true if $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$.

Clearly, Theorem 4.9.1 is equivalent to $(C, 1)$ -summability of the expression between the brackets in the left-hand side of (4.9.4).

Zeros of orthogonal polynomials are just eigenvalues of truncated Jacobi matrices. If g is a real valued $d\alpha$ -measurable function and all the moments of $g d\alpha$ are finite, then we can form the matrix $T(g, d\alpha) = \{a_{kj}\}$, $k, j = 0, 1, \dots$, defined by

$$a_{kj} = \int_{\mathbb{R}} p_k(d\alpha, t) p_j(d\alpha, t) g(t) d\alpha(t). \quad (4.9.5)$$

Such a matrix $T(g, d\alpha)$ is called a Toeplitz matrix corresponding to $d\alpha$ and generated by g . For $n = 1, 2, \dots$, the truncated matrix $T_n(g, d\alpha)$ is defined by $T_n(g, d\alpha) = \{a_{kj}\}$, $k, j = 0, 1, \dots, n-1$. Since $T_n(g, d\alpha)$ is symmetric, its eigenvalues $A_{kn}(g, d\alpha)$, $k = 1, 2, \dots, n$, are all real. If $g(t) \equiv t$, then $A_{kn}(g, d\alpha)$ are precisely the zeros of $p_n(d\alpha)$.

It was Szegő [GrSz] who first investigated the eigenvalue distribution of such Toeplitz matrices generated by continuous functions when the measure $d\alpha$ satisfies Szegő's condition $\log \alpha'(\cos \theta) \in L_1$ in $[0, \pi]$. It turns out that both conditions on g and $d\alpha$ can be relaxed. This I first proved in [Ne27] by analytic means, and later reproved jointly with Máté and Totik in [MáNeTo1] by more conventional matrix-theoretical methods. It brings me great pleasure that both proofs use Christoffel functions in a nontrivial way. The result I am talking about is the following.

THEOREM 4.9.4 [Ne27]. *Let the measure $d\alpha$ be such that $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$. Assume that the Toeplitz matrix $T(g, d\alpha)$ is generated by a function $g \in L_\infty(d\alpha)$. Let G be a continuous function in an interval containing the essential range of g . Then the eigenvalues $A_{kn}(g, d\alpha)$ of the truncated matrix $T_n(g, d\alpha)$ satisfy*

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n G(A_{kn}(g, d\alpha)) = \pi^{-1} \int_{-1}^1 G(g(t))(1-t^2)^{-1/2} dt. \quad (4.9.6)$$

The analytic proof of Theorem 4.9.4 is based on the following

THEOREM 4.9.5 [Ne27]. *Let $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ a.e. in $[-1, 1]$. Suppose that $f \in L_\infty$ in the square $[-1, 1] \times [-1, 1]$ and satisfies*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_x^{x+\varepsilon} |f(x, t) - f(x, x)| dt = 0 \quad (4.9.7)$$

for almost every $x \in [-1, 1]$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-1} \int_{-1}^1 \int_{-1}^1 K_n(d\alpha, x, t)^2 f(x, t) d\alpha(x) d\alpha(t) \\ = \pi^{-1} \int_{-1}^1 f(t, t)(1-t^2)^{-1/2} dt \end{aligned} \quad (4.9.8)$$

(cf. (3.12) regarding K_n).

I recommend that the reader compare this result with Theorem 4.5.4.

Proof of Theorem 4.9.4. First we notice that, by Lebesgue's theorem, the function $f(x, t) = G(g(t))$ satisfies (4.9.7). Thus, since

$$\begin{aligned} \int_{-1}^1 G(g(x)) K_n(d\alpha, x, x) d\alpha(x) \\ = \int_{-1}^1 \int_{-1}^1 G(g(t)) K_n(d\alpha, x, t)^2 d\alpha(x) d\alpha(t), \end{aligned} \quad (4.9.9)$$

The formula (4.9.6) will be proved if we can show the validity of

$$\lim_{n \rightarrow \infty} \left[n^{-1} \sum_{k=1}^n G(A_{kn}) - n^{-1} \int_{-1}^1 G(g(x)) K_n(d\alpha, x, x) d\alpha(x) \right] = 0. \quad (4.9.10)$$

For every n , we take a system of orthogonal eigenvectors of $T_n(g, d\alpha)$, say, $\mathbf{e}_k = (e_{k0}, e_{k1}, \dots, e_{k,n-1})$, $k = 1, 2, \dots, n$, and we construct n polynomials

$\psi_{kn} = (\mathbf{e}_k, \mathbf{P}_n(d\alpha))$, $k = 1, 2, \dots, n$, where $\mathbf{P}_n(d\alpha) = (p_0(d\alpha), p_1(d\alpha), \dots, p_{n-1}(d\alpha))$. It is easy to see that these polynomials ψ_{kn} satisfy

$$\int_{-1}^1 \psi_{kn}(x) \psi_{jn}(x) d\alpha(x) = \delta_{kj}, \quad (4.9.11)$$

$$\int_{-1}^1 \psi_{kn}(x) \psi_{jn}(x) g(x) d\alpha(x) = A_{kn}(g, d\alpha) \delta_{kj} \quad (4.9.12)$$

and

$$\sum_{k=1}^n \psi_{kn}(x) \psi_{kn}(t) = K_n(d\alpha, x, t). \quad (4.9.13)$$

In view of (4.9.13), we have

$$\begin{aligned} n^{-1} \sum_{k=1}^n G(A_{kn}) - n^{-1} \int_{-1}^1 G(g(x)) K_n(d\alpha, x, x) d\alpha(x) \\ = n^{-1} \sum_{k=1}^n \int_{-1}^1 [G(A_{kn}) - G(g(x))] \psi_{kn}(x)^2 d\alpha(x) \equiv \mathbb{0}. \end{aligned} \quad (4.9.14)$$

Now fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|G(x) - G(y)| < \varepsilon$ for $|x - y| < \delta$. Then we can write

$$\begin{aligned} \mathbb{0} &\equiv n^{-1} \sum_{k=1}^n \int_{|A_{kn} - g(x)| < \delta} [G(A_{kn}) - G(g(x))] \psi_{kn}(x)^2 d\alpha(x) \\ &+ n^{-1} \sum_{k=1}^n \int_{|A_{kn} - g(x)| \geq \delta} [G(A_{kn}) - G(g(x))] \psi_{kn}(x)^2 d\alpha(x) \equiv \mathbb{0}_1 + \mathbb{0}_2. \end{aligned} \quad (4.9.15)$$

By the choice of δ ,

$$|\mathbb{0}_1| \leq \varepsilon n^{-1} \sum_{k=1}^n \int_{-1}^1 \psi_{kn}(x)^2 d\alpha(x) = \varepsilon. \quad (4.9.16)$$

We also have

$$|\mathbb{0}_2| \leq 2\delta^{-2} n^{-1} \max |G| \sum_{k=1}^n \int_{-1}^1 [A_{kn} - g(x)]^2 \psi_{kn}(x)^2 d\alpha(x). \quad (4.9.17)$$

Using (4.9.11)–(4.9.13), we can evaluate the right-hand side of (4.9.17) and obtain

$$|\mathbb{0}_2| < 2\delta^{-2} n^{-1} \max |G| \left[\int_{-1}^1 g(x)^2 K_n(d\alpha, x, x) d\alpha(x) - \sum_{k=1}^n A_{kn}^2 \right]. \quad (4.9.18)$$

Since

$$K_n(d\alpha, x, x) = \int_{-1}^1 K_n(d\alpha, x, t)^2 d\alpha(x) \quad (4.9.19)$$

and

$$\begin{aligned} \sum_{k=1}^n A_{kn}^2 &= \text{Trace}(T_n(g, d\alpha)^2) \\ &= \int_{-1}^1 \int_{-1}^1 g(x) g(t) K_n(d\alpha, x, t)^2 d\alpha(x) d\alpha(t), \end{aligned} \quad (4.9.20)$$

we can rewrite (4.9.18) in the form

$$\begin{aligned} \|\mathbb{I}_2\| &\leq 2\delta^{-2} n^{-1} \max |G| \int_{-1}^1 \int_{-1}^1 [g(x)^2 - g(x)g(t)] \\ &\quad \times K_n(d\alpha, x, t)^2 d\alpha(x) d\alpha(t). \end{aligned} \quad (4.9.21)$$

Note that the function $f(x, t) = g(x)g(t)$ satisfies (4.9.7). Therefore, by (4.9.8), $\|\mathbb{I}_2\| \rightarrow 0$ as $n \rightarrow \infty$. Taking (4.9.15) and (4.9.16) into account, we get $\limsup \|\mathbb{I}\| \leq \varepsilon$, as $n \rightarrow \infty$. Since $\varepsilon > 0$ is arbitrary, (4.9.10) follows (cf. (4.9.14)), and so does the theorem. ■

Because they are not directly related to orthogonal polynomials and Christoffel functions, I have not discussed extensions of Szegő's results such as problems associated with distribution of eigenvalues of Hermitian integral operators and so forth (cf. [Lan, LanWi, Wi1–Wi6, Wilf] and the references therein). In Section 4.18, I will return to zeros of orthogonal polynomials where the case of infinite intervals will be examined.

4.10. Hermite–Fejér Interpolation and Derivatives of Christoffel Functions

For given $d\alpha$, f and n , the Hermite–Fejér interpolation polynomial $H_n(d\alpha, f)$ is the unique polynomial of degree at most $2n - 1$ which satisfies the conditions

$$H_n(d\alpha, f, x_{kn}(d\alpha)) = f(x_{kn}(d\alpha)) \quad \text{and} \quad H_n'(d\alpha, f, x_{kn}(d\alpha)) = 0 \quad (4.10.1)$$

for $k = 1, 2, \dots, n$, where $x_{kn}(d\alpha)$ are the zeros of the orthogonal polynomials $p_n(d\alpha)$ (cf. (3.5)). We can express $H_n(d\alpha, f)$ in terms of the fundamental polynomials of Lagrange interpolation $l_{kn}(d\alpha)$ (cf. (3.16)) as

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) [1 - 2l'_{kn}(d\alpha, x_{kn})(x - x_{kn})] l_{kn}(d\alpha, x)^2. \quad (4.10.2)$$

Let us compute $l'_{kn}(d\alpha, x_{kn})$. We have, by the trace invariance formula,

$$\lambda_{kn}(d\alpha, x)^{-1} = \sum_{k=1}^n l_{kn}(d\alpha, x)^2 \lambda_{kn}(d\alpha)^{-1} \quad (4.10.3)$$

(see, e.g., [Fr31b, p. 25]), and, differentiating both sides, we obtain

$$-\lambda'_n(d\alpha, x) \lambda_n(d\alpha, x)^{-2} = \sum_{k=1}^n 2l'_{kn}(d\alpha, x) l_{kn}(d\alpha, x) \lambda_{kn}(d\alpha)^{-1}. \quad (4.10.4)$$

Putting $x = x_{kn}$ leads to

$$\lambda'_n(d\alpha, x_{kn}) \lambda_{kn}(d\alpha)^{-1} = -2l'_{kn}(d\alpha, x_{kn}). \quad (4.10.5)$$

Substituting (4.10.5) into (4.10.2), we get

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) [1 + \lambda_{kn}(d\alpha)^{-1} \lambda'_n(d\alpha, x_{kn})(x - x_{kn})] l_{kn}(d\alpha, x)^2. \quad (4.10.6)$$

This is Freud's representation of the Hermite–Fejér interpolating polynomials in terms of the Christoffel function [Fr9]. It turns out that (4.10.6) is much more convenient to handle than the standard representation (4.10.2), which is sometimes written as

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) [1 - p''_n(d\alpha, x_{kn}) p'_n(d\alpha, x_{kn})^{-1} (x - x_{kn})] l_{kn}(d\alpha, x)^2. \quad (4.10.7)$$

Naturally, when one investigates Hermite–Fejér interpolation based at zeros of classical orthogonal polynomials such as Jacobi, Hermite, and Laguerre polynomials, there is no dispute as to the usefulness of (4.10.7) since the second-order differential equation satisfied by these polynomials yields immediately a convenient expression for

$$p''_n(d\alpha, x_{kn}) p'_n(d\alpha, x_{kn}) = -2l'_{kn}(d\alpha, x_{kn}) \quad (4.10.8)$$

which enables one to proceed with suitable estimates leading to convergence of these polynomials to f . In the general case, however, we cannot count on differential equations, or for that matter on anything such as generating functions, integral representations or difference equations, and thus one tries to avoid dealing with second derivatives of orthogonal polynomials, especially since one can hardly negotiate the polynomials themselves.

The realization that (4.10.5) and (4.10.6) hold should be counted as one

of Freud's seminal contributions towards orthogonal polynomials whose significance should not be underestimated.

As far as I am concerned, I do not believe that Hermite–Fejér interpolation deserves the popularity it has received in the past 60 years. Although, for any practical purpose, there are endlessly many papers dealing with convergence and/or divergence of Hermite–Fejér interpolation, most of these papers are based upon elegant identities resulting from the specific choice of interpolation nodes. Even when the nodes of interpolation are chosen to be zeros of orthogonal polynomials, most of the published research deals with classical orthogonal polynomials and pointwise convergence and/or divergence. The only exception is given by four papers of Freud [Fr9, Fr45, Fr46, Fr72], where he treats pointwise convergence of Hermite–Fejér interpolation taken at zeros of general orthogonal polynomial systems, and my recent joint papers with P. Vértesi [NeVé1, NeVé2], where we investigate weighted mean convergence of Hermite–Fejér interpolation at the zeros of generalized Jacobi polynomials. I do have another favored paper on Hermite–Fejér interpolation though, written by P. Vértesi [Vé], where necessary and sufficient conditions are given for convergence of Hermite–Fejér interpolation in terms of structural properties of functions and the behavior of the Hermite–Fejér interpolation polynomials at two points.

Freud did not simply observe (4.10.6); in [Fr9] he actually worked out a method for estimating the derivatives of the Christoffel functions. This method is simple and straightforward, and it consists of estimating the reciprocal of the Christoffel function with the aid of the extremum property (4.1.1). Since $\lambda_n(dx)^{-1}$ is a polynomial of degree at most $2n - 2$, one can apply either Bernstein's or Markov's inequality to estimate $[\lambda_n(dx)^{-1}]'$. Now $[\lambda_n(dx)^{-1}]' = -\lambda'_n(dx)[\lambda_n(dx)^{-2}]$, and thus two-sided estimates of $\lambda_n(dx)^{-1}$ and upper estimates of $[\lambda_n(dx)^{-1}]'$ yield the required estimates for $\lambda'_n(dx)$. No matter how unsophisticated this approach is, it provides deep results. For example, in [Fr9, Theorem 1] Freud proved the following

THEOREM 4.10.1 [Fr9]. *Let dx be absolutely continuous with support in $[-1, 1]$. Let $w = \alpha'$ be continuous and positive in $[-1, 1]$, and assume that, in a subinterval $[a, b] \subset (-1, 1)$, w satisfies the Dini–Lipschitz condition*

$$w(t) - w(y) = o(|\log |t - y||^{-1}), \quad a \leq t, y \leq b, \quad (4.10.9)$$

whereas the sequence $\{p_n(dx, x)\}$, $n = 1, 2, \dots$, is uniformly bounded in $[a, b]$. Let f be bounded in $[-1, 1]$ and continuous at -1 and 1 . If f is continuous at x ($a < x < b$), then

$$\lim_{n \rightarrow \infty} H_n(dx, f, x) = f(x). \quad (4.10.10)$$

If f is continuous in $[a, b]$, then (4.10.10) holds uniformly on every fixed closed subinterval of (a, b) .

It is natural for the reader to wonder why it is necessary to assume, in Theorem 4.10.1, that f is continuous at -1 and 1 when the action takes place inside $[a, b]$. Well, the reason is that the above method of Freud does not yield sufficiently sharp estimates for the derivatives $\lambda'_n(d\alpha)$ of the Christoffel function. As a matter of fact, in [Fr9, formula (48)], Freud can only prove

$$\lambda'_n(d\alpha, x) = O(1), \quad n = 1, 2, \dots, \quad (4.10.11)$$

uniformly for $-1 \leq x \leq 1$, if $d\alpha$ satisfies the conditions of the theorem, and in order to be able to remove the requirement that f be continuous at the endpoints of $[-1, 1]$, one needs to show

$$\lambda'_n(d\alpha, x) = O(1/n)(1-x^2)^{-1/2}, \quad n = 1, 2, \dots, \quad (4.10.12)$$

uniformly for $-1 + n^{-2} \leq x \leq 1 - n^{-2}$. It turns out that the latter needs much more sophisticated arguments.

Another way of looking at $\lambda'_n(d\alpha)$ is based on (4.1.1) and amounts to comparing the derivatives of Christoffel functions of two different measures provided that we know how the two measures are related to each other. Theorem 4.5.8 shows how to do this for the Christoffel functions, and thus there should be no reason to expect that this would be impossible to achieve for the derivatives of the Christoffel functions as well. The basic idea is contained in the following theorem, which was proved in [NeVé2, Lemma 1, p. 31].

THEOREM 4.10.2 [NeVé2]. *Let Δ be a fixed interval. Let g be a positive continuous function in Δ such that g is differentiable on some set $D \subset \Delta$ and both $\sup |g'(x)|$ when $x \in D$ and $\sup |g(x) - g(t) - g'(x)(x-t)|(x-t)^{-2}$ when $x \in D$ and $t \in \Delta$ are finite. Let $d\alpha$ be supported in Δ and let $d\beta$ be defined by*

$$d\beta = g d\alpha. \quad (4.10.13)$$

Then

$$\begin{aligned} & |g(x)[\lambda_n(d\beta, x)^{-1}]' - [\lambda_n(d\alpha, x)^{-1}]'| \\ & \leq K \sum_{k=n-2}^n [|p_k(d\alpha, x)| + |p'_k(d\alpha, x)|], \end{aligned} \quad (4.10.14)$$

uniformly for $x \in D$ and $n = 1, 2, \dots$, where K is a fixed constant.

In other words, if one has sufficient information regarding $[\lambda_n(d\alpha, x)^{-1}]'$, then one is able to say a fair amount regarding $[\lambda_n(d\beta, x)^{-1}]'$. The proof of Theorem 4.10.2 is based on the identity

$$\begin{aligned} & g(x)[\lambda_n(d\beta, x)^{-1}]' - [\lambda_n(d\alpha, x)^{-1}]' \\ &= 2g'(x) \int_A K_n(d\beta, x, t)(x-t)[\partial K_n(d\alpha, x, t)/\partial x] d\alpha(t) \\ &\quad + 2 \int_A K_n(d\beta, x, t)[\partial K_n(d\alpha, x, t)/\partial x] \\ &\quad \times [g(x) - g(t) - g'(x)(x-t)] d\alpha(t), \end{aligned} \tag{4.10.15}$$

which can be proved by direct verification. Since one of my goals is to stay at the conceptual level and not immerse myself in unpleasant computations, I refrain from going into the details of proving Theorem 4.10.2 using (4.10.15). Instead, I point out that, using (4.10.14) with the Lebesgue measure (i.e., with Legendre polynomials), one can easily prove (4.10.12), which leads to the following result of S. S. Bonan.

THEOREM 4.10.3 [Bo1]. *Theorem 4.10.1 remains true for bounded functions f which are not necessarily continuous at the endpoints of $[-1, 1]$.*

Seventeen years after publishing [Fr9], Freud returned to the problem of convergence of Hermite–Fejér interpolation in [Fr46], which I consider one of his masterpieces. As before, the main emphasis is on estimating the derivative of the Christoffel functions. I do not know how, but he came up with the wonderful idea that if a weight function (i.e., α') is monotonic, then so is the corresponding Christoffel function. More precisely, in [Fr46, Lemma 1, p. 308] Freud proves the following

THEOREM 4.10.4 [Fr46]. *Let $d\alpha$ be supported on the positive real line, and suppose that it is absolutely continuous. If, for some real r , $x^r \alpha'(x)$ is a nonincreasing function, then $x^{r-1} \lambda_n(d\alpha, x)$, $n = 1, 2, \dots$, are all decreasing functions for $x > 0$.*

Proof of Theorem 4.10.4. Let $u > 1$. Then $(ux)^r \alpha'(ux) \leq x^r \alpha'(x)$ for $x \in \mathbb{R}$ so that

$$d\beta_u \leq d\alpha, \quad \text{where } \beta'_u(x) = u^r \alpha'(ux). \tag{4.10.15}$$

Thus by the extremal property (4.1.1),

$$\lambda_n(d\beta_u, x) \leq \lambda_n(d\alpha, x). \tag{4.10.16}$$

It is a matter of simple computation to show that

$$\lambda_n(d\beta_u, x) = u'^{-1} \lambda_n(d\alpha, ux), \quad (4.10.17)$$

and thus the theorem follows from (4.10.16) and (4.10.17). ■

Following standard practice, let us define the linear functions v_{kn} by

$$v_{kn}(d\alpha, x) = [1 - 2l'_{kn}(d\alpha, x_{kn})(x - x_{kn})], \quad k = 1, 2, \dots, n. \quad (4.10.18)$$

Hence

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{kn}) v_{kn}(d\alpha, x) l_{kn}(d\alpha, x)^2, \quad (4.10.19)$$

and whether or not H_n is a positive or bounded operator mostly depends on the properties of the functions (4.10.18). Moreover, in view of the linearity of the functions $v_{kn}(d\alpha)$, their positivity needs to be checked only at endpoints of the smallest interval containing the support of the measure $d\alpha$. Freud's identity (4.10.5) and Theorem 4.10.4 can be combined to prove the following unexpectedly simple and charming result of Freud in [Fr46, Lemma 2, p. 308].

THEOREM 4.10.5 [Fr46]. *Let $d\alpha$ be absolutely continuous with support in $[-1, 1]$. Assume that there are two numbers a and b such that $(1-x)^a \alpha'(x)$ is nondecreasing and $(1+x)^b \alpha'(x)$ is nonincreasing. Then we have*

$$v_{kn}(d\alpha, 1) \geq a \quad \text{and} \quad v_{kn}(d\alpha, -1) \geq b, \quad k = 1, 2, \dots, n, \quad (4.10.20)$$

for all $n = 1, 2, \dots$

Proof of Theorem 4.10.5. By Theorem 4.10.4, the function $(1-x)^{a-1} \lambda_n(d\alpha, x)$ increases in $[-1, 1]$ whereas $(1+x)^{b-1} \lambda_n(d\alpha, x)$ decreases. By differentiation one obtains

$$1 + (1-t) \lambda'_n(d\alpha, x) \lambda_n(d\alpha, x)^{-1} \geq a \quad (4.10.21)$$

and

$$1 - (1+t) \lambda'_n(d\alpha, x) \lambda_n(d\alpha, x)^{-1} \geq b \quad (4.10.22)$$

for $-1 \leq t \leq 1$. Now (4.10.20) follows from (4.10.5). ■

On the basis of Theorem 4.10.5, Freud [Fr46, Theorem 1, p. 312] then proves the following result, which is one of the very few genuinely first-rate theorems on convergence of Hermite–Fejér interpolation.

THEOREM 4.10.6 [Fr46]. *Let $d\alpha$ be absolutely continuous with support in $[-1, 1]$. Assume that there are two numbers a and b such that $(1-x)^a \alpha'(x)$*

is nondecreasing and $(1+x)^b \alpha'(x)$ is nonincreasing. Let the function f be bounded in $[-1, 1]$. Then

$$\lim_{n \rightarrow \infty} H_n(d\alpha, f, x) = f(x) \quad (4.10.23)$$

if f is continuous at $x \in (-1, 1)$, and (4.10.23) holds uniformly on any closed subinterval of $(-1, 1)$ where f is continuous.

Another gem is [Fr45], where Freud explains why Hermite–Fejér interpolation diverges for so many weight functions. Since no Christoffel functions are involved in his short and conceptual proof, I will not discuss that paper here.

This is what Freud *did* succeed in proving on Hermite–Fejér interpolation. The next question concerns what Freud did *not* do in relation to this interpolation process. Besides not dealing with routine problems, he completely missed weighted mean convergence of Hermite–Fejér interpolation, which is a natural question since

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} H_n(d\alpha, f, x) d\alpha(x) = \int_{\mathbb{R}} f(x) d\alpha(x) \quad (4.10.24)$$

whenever, say, the measure has a compact support and the function f is Riemann–Stieltjes integrable (cf. [Fr31b, p. 89]). The latter holds, of course, because (4.10.24) is equivalent to the convergence of the Gauss–Jacobi quadrature process.

Another question is *why* Freud missed investigating weighted mean convergence of Hermite–Fejér interpolation. For me the answer is clear: he did not possess the tools necessary for such an investigation. As it turns out, the tools come from Lagrange interpolation, and the connection is given by the identity which we found in [NeVé2, formula (85), p. 55],

$$H_n(d\alpha, f, x) = \sum_{k=1}^n f(x_{k_n}) l_{k_n}(d\alpha, x)^2 + a_n p_n(d\alpha, x) L_n(d\alpha, f \lambda'_n(d\alpha) p_{n-1}(d\alpha), x), \quad (4.10.26)$$

where a_n is the recurrence coefficient in (3.8). Naturally, the expert eye will immediately realize that this identity is a simple consequence of Freud's formula (4.10.5) and other identities involving orthogonal polynomials and Lagrange interpolation (cf. [Fr31b, Chap. 1]).

For mean convergence of Lagrange interpolation $C \cap L_2(d\alpha)$ is the natural space (cf. Erdős and Turán's [ErTu1]), and therefore formula (4.10.26) suggests that for Hermite–Fejér interpolation $C \cap L_1(d\alpha)$ is the right setting, if such a space exists at all. The other message in (4.10.26) is

that investigation of Lagrange interpolation and derivatives of Christoffel functions together (cf. Theorem 4.10.2) will necessarily lead to the right results regarding mean convergence of Hermite–Fejér interpolation. This philosophy was carried out in [NeVé2], where we systematically studied such problems for generalized Jacobi weight functions.

In this section, we define generalized Jacobi weights as follows. Let g be a positive continuous function in $[-1, 1]$ such that $g' \in \text{Lip } 1$. If w can be expressed in the form

$$w(x) = g(x)(1-x)^a(1+x)^b, \quad -1 \leq x \leq 1, \quad (4.10.27)$$

where $a > -1$, $b > -1$, then w is a smooth generalized Jacobi weight. A typical result is the following

THEOREM 4.10.7 [NeVé2]. *Let $d\alpha$ be absolutely continuous with support in $[-1, 1]$, and let α' be a smooth generalized Jacobi weight. Let $p > 0$, and let u and v be two Jacobi weight functions. Then*

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |f(t) - H_n(d\alpha, f, t)|^p u(t) dt = 0 \quad (4.10.28)$$

for every continuous function f satisfying

$$|f(x)| \leq \text{const } v(x), \quad -1 \leq x \leq 1, \quad (4.10.29)$$

if and only if

$$\int_{-1}^1 \alpha'(t)^{-p} u(t) dt < \infty. \quad (4.10.30)$$

I conclude this section with a confession: it was Freud's representation (4.10.6) of the Hermite–Fejér interpolating polynomials in terms of the Christoffel function which led me to the idea of investigating Christoffel functions via the $G_n(d\alpha)$ operators defined by (4.5.6), and thus, in one sense or another, Freud is indirectly responsible for many of the results he did not prove himself. Let me elaborate on this. In [Ne19, p. 57] I recommended rewriting (4.10.6) as

$$\begin{aligned} & H_n(d\alpha, f, x) \\ &= \sum_{k=1}^n f(x_{kn}) [\lambda_{kn}(d\alpha) + \lambda'_n(d\alpha, x_{kn})(x - x_{kn})] l_{kn}(d\alpha, x)^2 \lambda_{kn}(d\alpha)^{-1}. \end{aligned} \quad (4.10.31)$$

The expression in brackets on the right-hand side is the linear Taylor approximation of $\lambda_n(d\alpha, x)$. If we replace the expression in brackets by the

Christoffel function, then we end up with a positive operator, say $F_n(d\alpha)$, defined by

$$F_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \sum_{k=1}^n f(x_{kn}) l_{kn}(d\alpha, x)^2 \lambda_{kn}(d\alpha)^{-1}. \quad (4.10.32)$$

It is easy to see that these rational functions also satisfy the interpolation property

$$F_n(d\alpha, f, x_{kn}(d\alpha)) = f(x_{kn}(d\alpha)) \quad \text{and} \quad F'_n(d\alpha, f, x_{kn}(d\alpha)) = 0, \quad (4.10.33)$$

and thus their behavior will be predictable, to say the least. I introduced this sequence of operators in [Ne19], where it enabled me to start investigations of what we now call generalized Szegő theory (cf. Section 4.13). Moreover, using the well-known formula

$$l_{kn}(d\alpha, x) = \lambda_{kn}(d\alpha) K_n(d\alpha, x_{kn}) \quad (4.10.34)$$

(cf. [Fr31b, formula (1.4.6), p. 25]), we can write (4.10.32) as

$$F_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \sum_{k=1}^n \lambda_{kn}(d\alpha) f(x_{kn}) K_n(d\alpha, x_{kn})^2, \quad (4.10.35)$$

which is the Gauss–Jacobi quadrature sum for $G_n(d\alpha)$ defined by (4.5.6). This is how I came to introduce the operators $G_n(d\alpha)$.

4.11. Szegő's Theory via Christoffel Functions

Szegő's theory concerns the behavior of complex orthogonal polynomials off the unit circle. It was first developed by Szegő [Sz4, Vol. I, pp. 69, 111; Sz4, Vol. I, p. 475] and S. N. Bernstein [Be3, Be4], and it was further enhanced, first by N. I. Akhiezer [Ak3], A. N. Kolmogorov [Ko], M. G. Krein [Kre1] and V. I. Smirnov [Sm], and then by Freud [Fr16, Fr17, Fr31a, b] and Geronimus [Ger2–Ger4]. The first significant simplification in solving Szegő's extremal problem (to be described shortly) was presented by him in [GrSz]. Besides [GrSz] the most popular book dealing with Szegő's theory is Freud's book [Fr31a, b], which devotes an entire chapter (Chap. 5) to Szegő's theory (a phrase coined by Freud). One of the unexpected fringe benefits of my recent work with Atti Máté and Vili Totik on extensions of Szegő's theory (which is valid under the assumption that $\log u' \in L_1$) to the case when $u' > 0$ almost everywhere was that the possibility of proving Szegő's results via considerations arising from the use of Christoffel functions emerged. This approach turns out to be simpler and more goal oriented than any other known attempt.

Szegő's (generalized) extremal problem consists of finding

$$\omega(d\mu, z) = \lim_{n \rightarrow \infty} \omega_n(d\mu, z), \quad |z| < 1 \quad (4.11.1)$$

($\omega(d\mu, z) = (2\pi)^{-1} \mu\{z\}$ for $|z| = 1$, and $\omega(d\mu, z) = 0$ for $|z| > 1$, which easily follows from the theory of moments, where $\mu\{z\}$ denotes the $d\mu$ -measure of the point z (cf. [Ak4])), and where the Christoffel function $\omega_n(d\mu)$ is defined by (3.19). For certain absolutely continuous measures, (4.11.1) was found by Szegő in [Sz4, Vol. I, p. 54] whereas the general case was treated by Kolmogorov [Ko], Krein [Kre1], and Smirnov [Sm]. The final touches on (4.11.1) were put on again by Szegő in [GrSz]. It was also Kolmogorov who associated Szegő's extremal problem with completeness of polynomials in $L_2(d\mu)$, and the latter turned out to be of crucial consequence in prediction theory. In what follows I describe the main results in Szegő's theory and also show how Christoffel functions can be used to prove them in an unexpectedly simple fashion.

THEOREM 4.11.1 [GrSz]. *For any measure $d\mu$ on the unit circle,*

$$\omega(d\mu, z) = (1 - |z|^2) |D(\mu', z)|^2, \quad |z| < 1. \quad (4.11.2)$$

Naturally, by Dini's theorem, the convergence in (4.11.1) is uniform on compact subsets of the open unit disk. The following measure-theoretic result enables one to reduce solution of problems such as (4.11.1) to solving them for absolutely continuous measures only. It was proved in [MáNeTo2], and it is an extension of a result of S. Kakutani (cf. [GrSz, Theorem 1.4]).

THEOREM 4.11.2 [MáNeTo2]. *Let ν be a finite positive Borel measure that is singular with respect to Lebesgue measure. Then there is a sequence $\{h_n\}$, $n = 1, 2, \dots$, of continuous functions on the real line such that*

$$0 < h_n(x) \leq 1 \quad (4.11.3)$$

for all x ,

$$\lim_{n \rightarrow \infty} h_n(x) = 1 \quad (4.11.4)$$

almost everywhere and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} h_n(t) \, d\nu(t) = 0. \quad (4.11.5)$$

If ν is confined to a finite interval and $T > 0$, then we may take each h_n to be periodic, with period T .

Proof of Theorem 4.11.1. First assume that $\log \mu' \in L_1$. Let $|z| < 1$. According to (4.1.2),

$$\omega_n(d\mu, z) = \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(z) = 1}} (2\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}. \quad (4.11.6)$$

It is fairly evident that on the right-hand side of (4.11.6) it is sufficient to consider such polynomials Π which do not vanish in the open unit disk (cf. [GrSz, p. 40]). Let Π be such a polynomial of degree $n-1$. Then the Szegő function $D(\mu' |\Pi|^2)$ is in H_2 (cf. (3.25)), and thus, applying Schwarz' inequality to the Taylor expansion of $D(\mu' \Pi)$, we obtain

$$|D(\mu' |\Pi|^2, z)|^2 \leq (1 - |z|^2)^{-1} (2\pi)^{-1} \int_0^{2\pi} |D(\mu' \Pi, u)|^2 dt, \quad u = e^{it}. \quad (4.11.7)$$

We have $|D(|\Pi|^2, z)| = |\Pi(z)|$ since $\Pi \neq 0$ in the open unit disk, and thus, by (3.25), (3.27), and (4.11.7),

$$|D(\mu', z)|^2 \leq (1 - |z|^2)^{-1} |\Pi(z)|^{-2} (2\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}, \quad (4.11.8)$$

from which

$$\omega_n(d\mu, z) \geq (1 - |z|^2) |D(\mu', z)|^2, \quad |z| < 1, \quad (4.11.9)$$

follows immediately. Since $\{\omega_n(d\mu, z)\}$ is a decreasing sequence, (4.11.1) obviously exists, and, passing to the limit in (4.11.9), we obtain

$$\omega(d\mu, z) \geq (1 - |z|^2) |D(\mu', z)|^2, \quad |z| < 1. \quad (4.11.10)$$

Now we concentrate on proving the opposite inequality. It follows from (4.11.1) and (4.11.6) that, for every polynomial Π ,

$$\omega(d\mu, z) \leq |\Pi(z)|^{-2} (2\pi)^{-1} \int_0^{2\pi} |\Pi(u)|^2 d\mu(t), \quad u = e^{it}. \quad (4.11.11)$$

By Theorem 4.11.2, there is a sequence $\{h_n\}$, $n = 1, 2, \dots$, of continuous 2π -periodic functions such that (4.11.3)–(4.11.5) hold. For given z ($|z| < 1$), $N = 1, 2, \dots$, $\varepsilon > 0$, $m = 1, 2, \dots$, $n = 1, 2, \dots$, and $M = 1, 2, \dots$, let Π be a polynomial such that $\Pi \neq 0$ in the open unit disk and

$$|\Pi(e^{it})| = \sigma_M((h_n \sigma_m(1/(\varepsilon + \mu')) |K_N(dt, e^{it}, z)|^2)^{1/2}, t), \quad (4.11.12)$$

where σ_k denotes the arithmetic means (i.e., Fejér sums) of the trigonometric Fourier series of the functions under consideration and $dl(t) = dt$ refers to the Lebesgue measure on $[0, 2\pi]$. Since $\Pi \neq 0$ in the open unit disk, we have $|D(|\Pi|, z)|^2 = |\Pi(z)|$. The next step is to substitute (4.11.12) back into (4.11.11) and then let, first, $M \rightarrow \infty$, then $n \rightarrow \infty$, then $m \rightarrow \infty$, and then $\varepsilon \rightarrow 0$. All limiting procedures are justified by Lebesgue's Bounded and Monotone Convergence Theorems, and we obtain

$$\begin{aligned} \omega(d\mu, z) &\leq |D(\mu', z)|^2 |K_N(dl, z, z)|^{-2} (2\pi)^{-1} \\ &\quad \times \int_0^{2\pi} |K_N(dl, u, z)|^2 dt, \quad u = e^{it}, \end{aligned} \quad (4.11.13)$$

that is

$$\omega(d\mu, z) \leq |D(\mu', z)|^2 \omega_N(dl, z) \quad (4.11.14)$$

for every $N = 1, 2, \dots$. Since the orthonormal polynomials associated with the Lebesgue measure are $z^n, n = 0, 1, 2, \dots$, one has no problem in evaluating the right-hand side of (4.11.14), and, letting $N \rightarrow \infty$, we arrive at

$$\omega(d\mu, z) \leq (1 - |z|^2) |D(\mu', z)|^2, \quad |z| < 1, \quad (4.11.15)$$

which, together with inequality (4.11.10), completes the proof of Theorem 4.11.1 when $\log \mu' \in L_1$. Otherwise, we apply (4.11.2) with $d\mu_\delta$, where $d\mu_\delta = d\mu + \delta dl$ ($\delta > 0$, dl denotes the Lebesgue measure), and then let $\delta \searrow 0$, which proves (4.11.2) in the general case as well. ■

Now Szegő's theory can be summarized by the following

THEOREM 4.11.3 [GrSz]. *Let $\log \mu' \in L_1$. Then*

$$\lim_{n \rightarrow \infty} \kappa_n(d\mu) = D(\mu', 0)^{-1}, \quad (4.11.16)$$

$$\lim_{n \rightarrow \infty} \varphi_n^*(d\mu, z) = D(\mu', z)^{-1}, \quad |z| < 1, \quad (4.11.17)$$

$$\sum_{k=0}^{\infty} \overline{\varphi_k(d\mu, z)} \varphi_k(d\mu, u) = (1 - \bar{z}u)^{-1} \overline{D(\mu', z)^{-1}} D(\mu', u)^{-1}, \quad |z|, |u| < 1, \quad (4.11.18)$$

and

$$\lim_{n \rightarrow \infty} \varphi_n(d\mu, z) = 0, \quad |z| < 1. \quad (4.11.19)$$

The convergence in (4.11.17)–(4.11.19) is uniform on compact subsets of the open unit disk. Moreover,

$$\lim_{n \rightarrow \infty} z^{-n} \varphi_n(d\mu, z) = \bar{D}(\mu', z^{-1})^{-1}, \quad |z| > 1, \quad (4.11.20)$$

uniformly on compact sets in the domain $|z| > 1$ on the Riemann sphere.

Proof of Theorem 4.11.3. Applying Szegő's Christoffel–Darboux formula (3.21) with $z = u = 0$, we obtain

$$\kappa_n(d\mu) = \omega_n(d\mu, 0)^{-1/2} \quad (4.11.21)$$

(cf. (3.19) and (3.20)) so that (4.11.16) is equivalent to the case $z = 0$ in Theorem 4.11.1. The next step is to prove (4.11.19). It follows from (4.11.2) that

$$\sum_{k=0}^{\infty} |\varphi_k(d\mu, z)|^2 = (1 - |z|^2)^{-1} |D(\mu', z)|^{-2}, \quad |z| < 1 \quad (4.11.22)$$

(cf. (4.11.1) and (3.20)), and thus (4.11.19) holds for every z in the open unit disk. Moreover, by (4.11.22), the sequence of orthogonal polynomials $\{\varphi_n(d\mu, z)\}$ is uniformly bounded on compact subsets of the open unit disk, which implies uniform convergence in (4.11.19) on compact subsets of the open unit disk. Now we are in a position to verify (4.11.17). By Szegő's formula (3.21) applied with $z = u$, we have

$$(1 - |z|^2) K_n(d\mu, z, z) = |\varphi_n^*(d\mu, z)|^2 - |\varphi_n(d\mu, z)|^2, \quad (4.11.23)$$

that is, by (3.19),

$$(1 - |z|^2) = \omega_n(d\mu, z)^{-1} |\varphi_n^*(d\mu, z)|^2 - \omega_n(d\mu, z)^{-1} |\varphi_n(d\mu, z)|^2. \quad (4.11.24)$$

By (4.11.19),

$$\lim_{n \rightarrow \infty} \omega_n(d\mu, z)^{-1} |\varphi_n(d\mu, z)|^2 = 0, \quad |z| < 1, \quad (4.11.25)$$

and thus

$$\lim_{n \rightarrow \infty} \omega_n(d\mu, z)^{-1} |\varphi_n^*(d\mu, z)|^2 = (1 - |z|^2), \quad |z| < 1. \quad (4.11.26)$$

Now (4.11.26), combined with (4.11.2), yields

$$\lim_{n \rightarrow \infty} |\varphi_n^*(d\mu, z)| = |D(\mu', z)|^{-1}, \quad |z| < 1. \quad (4.11.27)$$

Since

$$\varphi_n^*(d\mu, 0) = \kappa_n(d\mu) > 0 \quad \text{and} \quad D(\mu', 0) > 0 \quad (4.11.28)$$

(cf. (3.17), (3.22), and (3.25)), and we have already proved (4.11.16), formula (4.11.17) follows from (4.11.27). Formula (4.11.18) is a direct consequence of (4.11.17), (4.11.19), and Szegő's summation formula (3.22). Finally, (4.11.20) is equivalent to (4.11.17) (cf. the *-transformation defined by (3.22)). Thus we have succeeded in proving the main results of Szegő's theory by using Christoffel functions. ■

4.12. *Asymptotics for Orthogonal Polynomials and Equiconvergence of Orthogonal Fourier Series*

Freud made two lasting contributions to the theory of orthogonal polynomials on the unit circle. The first provides asymptotics for the orthogonal polynomials on the circle itself under conditions substantially weaker than those assumed by Bernstein [Be2, Be4], Szegő [Sz2], and Geronimus [Ger2–Ger4]. The second is related to convergence of orthogonal Fourier series, and improves upon Szegő's theorem on equiconvergence of those series with trigonometric Fourier series. The idea of reducing problems of convergence of orthogonal series to that of trigonometric Fourier series was first developed by A. Haar [Ha] in 1917 and it did indeed simplify finding convergence (and summability) conditions for Fourier series in orthogonal polynomials. Before going into detail regarding equiconvergence of orthogonal Fourier series, I will briefly report on Freud's results concerning asymptotics of orthogonal polynomials (cf. [Fr16, Fr17, Fr31a, b]).

To my great regret, at the present time I cannot (and neither could Freud) prove these asymptotic formulas via the exclusive use of Christoffel functions. Instead, the main tool of the trade is Szegő's observation that the *-transforms of the orthogonal polynomials $\varphi_n(d\mu)$ are essentially nothing else but partial sums of orthogonal Fourier expansions of the Szegő function (cf. (3.25)). More precisely, it follows from (3.20) and Szegő's Christoffel–Darboux formula (3.21), applied with $u = 0$, that

$$\kappa_n(d\mu) \varphi_n^*(d\mu, z) = \sum_{k=0}^{n-1} \overline{\varphi_k(d\mu, 0)} \varphi_k(d\mu, z). \quad (4.12.1)$$

By Theorems 4.10.1 and 4.10.3,

$$D(\mu', 0)^{-1} D(\mu', z)^{-1} = \sum_{k=0}^{\infty} \overline{\varphi_k(d\mu, 0)} \varphi_k(d\mu, z) \quad (4.12.2)$$

in $L_2(d\mu)$, and thus, in view of (4.11.16), we have

THEOREM 4.12.1 [Sz2]. *If $\log \mu' \in L_1$, then*

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} |\varphi_n^*(d\mu, z) - D(\mu', z)^{-1}|^2 d\mu(t) = 0, \quad z = e^{it}, \quad (4.12.3)$$

which, in terms of the orthogonal polynomials themselves, can be written as

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} |\varphi_n(d\mu, z) - z^n \overline{D(\mu', z)^{-1}}|^2 d\mu(t) = 0, \quad z = e^{it}. \quad (4.12.4)$$

Pointwise versions of (4.12.4) naturally require analyzing conditions for pointwise convergence of the series in (4.12.2). What I find to be the strongest result so far concerning pointwise asymptotics is the following theorem of Freud, which was first published in his book on orthogonal polynomials [Fr31a, b], an unorthodox way to announce new results, indeed.

THEOREM 4.12.2 [Fr31a, b]. *Let $\log \mu' \in L_1$, and let $t \in [0, 2\pi]$ be fixed. Assume that $d\mu$ is absolutely continuous in a neighborhood Δ of t , $\mu' \in L_\infty$ and $(\mu')^{-1} \in L_\infty$ in Δ , and*

$$\int_{\Delta} |\mu'(t) - \mu'(y)|^2 |t - y|^{-2} dy < \infty. \quad (4.12.5)$$

Then

$$\lim_{n \rightarrow \infty} [\varphi_n(d\mu, z) - z^n \overline{D(\mu', z)^{-1}}] = 0, \quad z = e^{it}. \quad (4.12.6)$$

The question whether the asymptotic formula (4.12.6) can be differentiated seems to be more complicated. While there have been some efforts to obtain asymptotics for the derivatives of orthogonal polynomials (cf. [Gol2, Hör, Ra]), it was only recently that this could be achieved under conditions no more restrictive than those appearing in Freud's Theorem 4.12.2. For me this is a particularly pleasing circumstance since it was my paper [Ne23] which succeeded in removing the more restrictive conditions imposed on the measure in the above-mentioned papers.

THEOREM 4.12.3 [Ne23]. *Let $d\mu$ and t satisfy the conditions of Theorem 4.12.2. Then*

$$\lim_{n \rightarrow \infty} [n^{-k} \varphi_n^{(k)}(d\mu, z) - z^{n-k} \overline{D(\mu', z)^{-1}}] = 0, \quad z = e^{it}, \quad (4.12.7)$$

for every fixed positive integer k .

Now let us return to equiconvergence of orthogonal Fourier series. In his seminal paper [Ha], A. Haar proved that orthogonal Legendre series and Chebyshev series of integrable functions are equiconvergent; i.e., the difference of the corresponding appropriate partial sums converges to 0. Haar's proof itself is much less exciting than the idea of reducing convergence of one series to that of another one, and it is actually a careful analysis of the asymptotic formula for Legendre polynomials with sufficiently accurate remainder terms. In fact, Haar's method is directly applicable to all classical orthogonal polynomial series, such as Jacobi, Hermite, and Laguerre series (cf. [Sz2]). The real fun starts when one leaves the road covered by remnants of classical orthogonal polynomials and starts to examine general orthogonal polynomial series. Here the glory belongs to Szegő (cf. [Sz4, Vol. I, p. 437; Sz2]), whose results were later recast and generalized by J. Korous [Koro3–Koro5], Geronimus [Ger2], and Freud [Fr31a, b]. The strongest results available regarding equiconvergence of orthogonal Fourier series are found in [Ne19]. Naturally, having had the pleasure of standing on the shoulders of this distinguished company, my job of putting the pieces together and adding my expertise on Christoffel functions was more or less a logically unavoidable conclusion of approximately 60 years of research. Oh yes, my reader, it is the Christoffel function again which keeps the orthogonal Fourier series within the norms of mathematico-socially acceptable and expected behavior. It is somewhat unfortunate, however, that the technical details associated with equiconvergence of orthogonal Fourier series have not been crystallized yet to the extent that it can be presented without introducing elements of ugly mathematics, i.e., mathematics involving long chains of estimates and inequalities leading to the right place without providing a continuous flow of eye- and mind-pleasing landscapes. For this reason you and I, my reader, will take the easy way out, which consists of concentrating on the main ideas and leaving out much of the detail.

In the rest of this section we deal with measures supported on the real line and our object is to investigate equiconvergence of two orthogonal Fourier series

$$S(d\alpha, f) = \sum_{k=0}^{\infty} c_k(d\alpha, f) p_k(d\alpha) \quad (4.12.8)$$

and

$$S(d\beta, f) = \sum_{k=0}^{\infty} c_k(d\beta, f) p_k(d\beta), \quad (4.12.9)$$

where the Fourier coefficients c_k are defined by a formula similar to (3.11).

Equiconvergence of $S(d\alpha, f)$ and $S(d\beta, f)$ at a particular point x simply refers to the fact that

$$\lim_{n \rightarrow \infty} [S_n(d\alpha, f, x) - S_n(d\beta, f, x)] = 0, \quad (4.12.10)$$

where S_n is the n th partial sum of the infinite series (cf. (3.10)).

In what follows we assume that $d\alpha$ and $d\beta$ are related to each other by

$$d\beta = g \, d\alpha, \quad (4.12.11)$$

where $g (\geq 0) \in L_1(d\alpha)$. We will also need the G operators defined by (4.5.6), which we used extensively in Section 4.5 while finding asymptotics for Christoffel functions. For reference, these operators are given by

$$G_n(d\alpha, h, x) = \lambda_n(d\alpha, x) \int_{\mathbb{R}} h(t) K_n(d\alpha, x, t)^2 \, d\alpha(t) \quad (4.12.12)$$

for $h \in L_1(d\alpha)$. Here, of course, λ_n is the Christoffel function and K_n is the reproducing kernel (cf. formulas (3.3), (3.12), and (3.13)).

The fundamental idea behind equiconvergence of orthogonal Fourier series is given by the following theorem proved in [Ne19, Lemma 8.1, p. 147], which crystallizes Szegő's concepts introduced in [Sz4, Vol. I, p. 437].

THEOREM 4.12.4 [Ne19]. *Let $\text{supp}(d\alpha)$ be compact, $g \geq 0$, $g \in L_1(d\alpha)$ and $g^{-1} \in L_1(d\alpha)$. Let $d\beta$ be defined by (4.12.11) and assume that $f \in L_2(d\beta)$. Then*

$$\begin{aligned} & |S_n(d\beta, f, x) - \lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} S_n(d\alpha, fg, x)| \\ & \leq \|f\|_{d\beta, 2} \{ \lambda_n(d\beta, x)^{-1} [G_n(d\alpha, g^{-1}, x) G_n(d\alpha, g, x) - 1] \}^{1/2} \end{aligned} \quad (4.12.13)$$

for all real x and $n = 1, 2, \dots$, where $\|f\|_{d\beta, 2}$ denotes the $L_2(d\beta)$ norm of f .

Proof of Theorem 4.12.4. Let us denote the left-hand side of (4.12.13) by $R(x)$. Then, by (3.14),

$$R(x) = \int_{\mathbb{R}} f(t) g(t) [K_n(d\beta, x, t) - \lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} K_n(d\alpha, x, t)] \, d\alpha(t). \quad (4.12.14)$$

Applying Schwarz' inequality, we obtain

$$|R(x)|^2 \leq \{ \|f\|_{d\beta, 2} \}^2 K(x), \quad (4.12.15)$$

where

$$K(x) = \int_{\mathbb{R}} [K_n(d\beta, x, t) - \lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} K_n(d\alpha, x, t)]^2 d\beta(t). \quad (4.12.16)$$

Let us evaluate $K(x)$ by multiplying out the integrand and using properties of reproducing kernel functions. We have

$$\begin{aligned} K(x) &= \int_{\mathbb{R}} K_n(d\beta, x, t)^2 d\beta(t) \\ &\quad - 2\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} \int_{\mathbb{R}} K_n(d\beta, x, t) K_n(d\alpha, x, t) d\beta(t) \\ &\quad + \lambda_n(d\alpha, x)^2 \lambda_n(d\beta, x)^{-2} \int_{\mathbb{R}} K_n(d\alpha, x, t)^2 d\beta(t) \\ &= K_n(d\beta, x, x) - 2\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} K_n(d\alpha, x, x) \\ &\quad + \lambda_n(d\alpha, x)^2 \lambda_n(d\beta, x)^{-2} \int_{\mathbb{R}} K_n(d\alpha, x, t)^2 g(t) d\alpha(t). \quad (4.12.17) \end{aligned}$$

Taking (3.3), (3.12), and (4.12.12) into consideration, we can conclude

$$K(x) = \lambda_n(d\beta, x)^{-1} [\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} G_n(d\alpha, g, x) - 1]. \quad (4.12.18)$$

By Theorem 4.5.8 (cf.(4.5.14)), we have

$$\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} \leq G_n(d\alpha, g^{-1}, x) \quad (4.12.19)$$

and thus

$$K(x) \leq \lambda_n(d\beta, x)^{-1} [G_n(d\alpha, g^{-1}, x) G_n(d\alpha, g, x) - 1]. \quad (4.12.20)$$

Now the theorem follows from (4.12.15) and (4.12.20). ■

Having proved Theorem 4.12.4, let us try to digest what it says. For the convenience of the reader, I reproduce (4.12.13) as

$$\begin{aligned} &|S_n(d\beta, f, x) - \lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} S_n(d\alpha, fg, x)| \\ &\leq \|f\|_{d\beta, 2} \{ \lambda_n(d\beta, x)^{-1} [G_n(d\alpha, g^{-1}, x) G_n(d\alpha, g, x) - 1] \}^{1/2}. \end{aligned} \quad (4.12.21)$$

First, let us analyze the right-hand side of this inequality. By Theorem 4.5.4,

$$\lim_{n \rightarrow \infty} G_n(d\alpha, h, x) = h(x), \quad (4.12.22)$$

where either $h = g$ or $h = g^{-1}$, provided that g is continuous and $d\alpha$ satisfies some conditions. Moreover, by imposing somewhat stricter conditions on g and $d\alpha$, one can actually improve (4.12.22) to

$$\lim_{n \rightarrow \infty} G_n(d\alpha, h, x) = h(x) + O(1/n) \quad (4.12.23)$$

($h = g$ or $h = g^{-1}$), which, in turn, would guarantee the boundedness of the right-hand side of (4.12.21), since $\lambda_n(d\beta, x)^{-1} = O(n)$ under fairly mild conditions on $d\beta$ (cf. Theorem 4.5.2).

Now let us take a closer look at the left-hand side of (4.12.21). Intuitively, it is clear that $S_n(d\alpha, fg, x) - g(x) S_n(d\alpha, f, x)$ tends to 0, as $n \rightarrow \infty$, whenever g is reasonably smooth (we all know that equiconvergence takes place for smooth functions; moreover, it does so for fairly obvious reasons). In fact, it is not difficult to show the validity of

$$\lim_{n \rightarrow \infty} S_n(d\alpha, fg, x) - g(x) S_n(d\alpha, f, x) = 0 \quad (4.12.24)$$

under reasonably mild conditions imposed upon g and $d\alpha$. The other term on the left-hand side of (4.12.21) to be taken care of is $\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1}$. In view of (4.12.23) and the techniques discussed in Section 4.5 (cf. Theorems 4.5.4 and 4.5.8), one can indeed prove

$$\lambda_n(d\alpha, x) \lambda_n(d\beta, x)^{-1} = g(x)^{-1} + O(1/n) \quad (4.12.25)$$

whenever g is sufficiently smooth and $d\alpha$ satisfies some conditions.

By (4.12.24) and (4.12.25), one can show that the left-hand side of (4.12.21) is essentially the same as the expression $S_n(d\beta, f, x) - S_n(d\alpha, f, x)$, which was our original primary target. What is left is to formulate accurately the conditions which are needed to guarantee the validity of all the above-discussed estimates. This was accomplished in [Ne19, Chap. 8] where I proved the following theorem on equiconvergence of orthogonal Fourier series and Chebyshev series. We need to introduce a few definitions in order to formulate this result.

The Chebyshev measure will be denoted by dT , i.e., $dT(t) = v dt$, where

$$v(t) = (1 - t^2)^{-1/2} (|t| < 1) \quad \text{and} \quad v(t) = 0 (|t| \geq 1). \quad (4.12.26)$$

For a given modulus of continuity ω , the class $B(x, \omega)$ is defined as follows.

The function F belongs to $B(x, \omega)$ if and only if $F'(x)$ exists and

$$|F(t) - F(x) - F'(x)(t - x)| \leq C_x \omega(|t - x|) |t - x| \quad (4.12.27)$$

for $|t - x|$ small, where C_x does not depend on t .

THEOREM 4.12.5 [Ne19]. *Let $d\alpha$ satisfy $\text{supp}(d\alpha) = [-1, 1]$, $\log \alpha' \in L_1$, and suppose that there exists a polynomial Π such that $\Pi^2/\alpha' \in L_1$ in $[-1, 1]$. Let $x \in (-1, 1)$ and let $d\alpha$ be absolutely continuous in a neighborhood of x . Assume that $\alpha' \in B(x, \omega)$ with $\omega(t)/t \in L_1$ in $[0, 1]$ and $\alpha'(x) > 0$. Then, for every $f \in L_2(d\alpha)$, we have the equiconvergence*

$$\lim_{n \rightarrow \infty} [S_n(d\alpha, f, x) - S_n(dT, f1_\delta, x)] = 0, \quad (4.12.28)$$

where 1_δ is the characteristic function of the interval $[x - \delta, x + \delta]$ and $\delta > 0$ is a sufficiently small fixed number. If, instead of the given point x , all the conditions are uniformly satisfied in a neighborhood of a fixed interval $\Delta \subset (-1, 1)$, then (4.12.28) holds uniformly for $x \in \Delta$, where, this time, 1_δ denotes the characteristic function of a sufficiently small δ -neighborhood of Δ .

4.13. Stepping beyond Szegő's Theory

Szegő's theory takes care of orthogonal polynomials when $\log \mu'$ is integrable. Here I will tell the story of what is happening when this condition is replaced by the much weaker one, $\mu' > 0$ almost everywhere. Szegő's theory was essentially created by a single individual. The principal players of the new game are A. Máté, E. A. Rahmanov, V. Totik, and I.

Not counting Erdős and Turán's [ErTu3], other results regarding distribution of zeros of orthogonal polynomials, and related asymptotics, the first steps towards extending Szegő's theory to orthogonal polynomials when the corresponding measure does not satisfy Szegő's condition of logarithmic integrability were taken by Rahmanov [Rah1] and me in [Ne19, Ne20, Ne24]. One of the many equivalent ways of formulating Szegő's limit result (4.11.16) is that

$$\lim_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z) z^{-n} \overline{D(\mu', \bar{z})} - 1|^2 d\theta = 0, \quad z = e^{i\theta}, \quad (4.13.1)$$

whenever $\log \mu' \in L_1$ (cf. (3.25) for the definition of Szegő's function D). Rahmanov [Rah1] proved the following weak version of (4.13.1),

$$\lim_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} F(\theta) |\varphi_n(d\mu, z)|^2 d\mu(\theta) = (2\pi)^{-1} \int_0^{2\pi} F(\theta) d\theta, \quad z = e^{i\theta}, \quad (4.13.2)$$

for every continuous function F provided that $\mu' > 0$ almost everywhere in $[0, 2\pi]$, and he also claimed to have proved the following variants of (4.11.16) and (4.11.20),

$$\lim_{n \rightarrow \infty} \kappa_n(d\mu)/\kappa_{n-1}(d\mu) = 1 \quad (4.13.3)$$

and

$$\lim_{n \rightarrow \infty} \varphi_n(d\mu)/\varphi_{n-1}(d\mu) = z, \quad |z| \geq 1, \quad (4.13.4)$$

if $\mu' > 0$ a.e., which, among others, also implies Theorem 4.5.7. As indicated after Theorem 4.5.7, correct proofs of (4.13.3) and (4.13.4) were published in [Rah4], and a conceptually simpler proof of the latter two limit relations was given in [MáNeTo2]. I wish also to point out that, on the basis of (3.21) (applied with $z = 0$ and $u = 0$), it is an easy exercise to show the equivalence of (4.13.3) and

$$\lim_{k \rightarrow \infty} \varphi_n(d\mu, 0) = 0, \quad (4.13.5)$$

where $\varphi_n(d\mu)$ is the monic orthogonal polynomial (cf. (3.23)).

What I proposed in [Ne19, Ne24] amounts to regarding Szegő's theory as a theory describing the behavior of orthogonal polynomials and related quantities in terms of another system, the system corresponding to Lebesgue measure, and in terms of Szegő functions of ratios (of the absolutely continuous components) of the associated measures. Then I went one leap further by comparing two orthogonal polynomial systems when the corresponding measures $d\mu_1$ and $d\mu_2$ do not satisfy Szegő's condition of logarithmic integrability. More precisely, assuming that one does have appropriate information regarding $d\mu_1$ and the associated orthogonal polynomials, and that one does know that $d\mu_2$ can be expressed in terms of $d\mu_1$ as

$$d\mu_2 = g d\mu_1, \quad (4.13.6)$$

where g is a reasonably well behaved function, one can then deduce information regarding the orthogonal polynomials associated with $d\mu_2$. This is how I found asymptotics for the leading coefficients $\gamma_n(d\alpha)$ of the (real) orthogonal polynomials corresponding to the (absolutely continuous) measure $d\alpha$ given by

$$\alpha'(x) = \exp\{-(1-x^2)^{-1/2}\}, \quad -1 \leq x \leq 1, \quad (4.13.7)$$

which is perhaps the simplest measure not covered by Szegő's theory. In this example I used the Pollaczek polynomials [Pol1; Pol2; Pol3; Sz1; Sz2,

p. 392] as the comparison system, which is orthogonal with respect to the absolutely continuous measure $d\beta^{(a,b)}$ defined by

$$\beta^{(a,b)}(x) = 2 \exp\{t(a \cos t + b)/\sin t\} [1 + \exp\{\pi(a \cos t + b)/\sin t\}]^{-1}, \quad (4.13.8)$$

where a and b are real numbers with $|b| < a$, $x = \cos t$ and $0 \leq t \leq \pi$. The result proved in [Ne19, p. 83] is the asymptotics

$$\lim_{n \rightarrow \infty} \gamma_n(d\alpha) 2^{-n} n^{-1/(2\pi)} = \Gamma((\pi + 1)/(2\pi)) D(\beta^{(1/\pi, 0)}/\alpha', 0). \quad (4.13.9)$$

My methods in [Ne19, Ne24] did not allow me to consider sufficiently general measures in (4.13.6), and I was restricted to working with measures where the function g in (4.13.6) and its reciprocal were Riemann integrable.

The next (still lasting) breakthrough in extending Szegő's theory started with [MáNeTo7], where various strong and weak convergence properties of complex and real orthogonal polynomials were proved. One of the main tools in generalizing Szegő's theory is the following limit relation proved in [MáNeTo7, Theorem 2.1].

THEOREM 4.13.1 [MáNeTo7]. *If $\mu' > 0$ almost everywhere, then*

$$\lim_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} - 1]^2 d\theta = 0, \quad z = e^{i\theta}. \quad (4.13.10)$$

What is significant in this theorem is not only that it strengthens Rahmanov's weak asymptotics (4.13.2), but also that, in view of the boundary value property of Szegő's function $|D(\mu')|^2 = \mu'$ (cf. (3.27)), formula (4.13.10) provides the natural extension of Szegő's L_2 asymptotics (4.13.1) which forms the basis of Szegő's theory. Moreover, I find it rather extraordinary that not only Szegő and Freud missed discovering Theorem 4.13.1 but also Rahmanov, who put so much effort into proving the weaker (4.13.2). Those who are familiar with Rahmanov's proof of (4.13.2) in [Rah1] will recognize that our proof of (4.13.10) borrowed some ideas from [Rah1]. Before presenting the proof of (4.13.10), I state the following

THEOREM 4.13.2 [Rah1]. *For all measures $d\mu$ and for all 2π -periodic continuous functions F , the limit relation*

$$\lim_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} F(\theta) |\varphi_n(d\mu, z)|^{-2} d\theta = (2\pi)^{-1} \int_0^{2\pi} F(\theta) d\mu(\theta), \quad z = e^{i\theta}, \quad (4.13.11)$$

holds.

Proof of Theorem 4.13.2. If F is a trigonometric polynomial, then (4.13.11) holds since

$$(2\pi)^{-1} \int_0^{2\pi} F(\theta) |\varphi_n(d\mu, z)|^{-2} d\theta = (2\pi)^{-1} \int_0^{2\pi} F(\theta) d\mu(\theta), \quad z = e^{i\theta}, \quad (4.13.12)$$

for $n > \deg(F)$ (cf. [Fr31b, Theorem 5.2.2, p.198]). Otherwise, we use a straightforward approximation argument. ■

For reasons of historical justice, I mention that Theorem 4.13.2 is implicitly contained in both Bernstein's and Szegő's reasoning when proving the orthogonality of the so-called Bernstein–Szegő polynomials (cf. [Fr31b, Theorem 5.4.5, p. 224]). However, Rahmanov deserves full credit for the realization that it can be used in situations that neither Bernstein nor Szegő thought of. For reasons that go beyond purely sentimental ones, I consider (4.13.12) the complex analogue of the Gauss–Jacobi quadrature formula (3.4), and then Theorem 4.13.2 is the analogue of the theorem on convergence of the Gauss–Jacobi quadrature process for measures with compact support.

Proof of Theorem 4.13.1. We have

$$\begin{aligned} 0 &\leq (2\pi)^{-1} \int_0^{2\pi} [|\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} - 1]^2 d\theta \\ &= (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)|^2 \mu'(\theta) d\theta - \pi^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta + 1 \\ &\leq (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)|^2 d\mu(\theta) - \pi^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta + 1 \\ &= 2 - \pi^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta, \quad z = e^{i\theta}. \end{aligned} \quad (4.13.13)$$

Therefore it will be sufficient to prove that

$$\liminf_{n \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta \geq 1, \quad z = e^{i\theta}. \quad (4.13.14)$$

To see this, let f be an arbitrary 2π -periodic nonnegative continuous function. By Hölder's inequality applied to appropriate functions, we obtain

$$\begin{aligned}
& \left\{ (2\pi)^{-1} \int_0^{2\pi} (f(\theta) \mu'(\theta))^{1/4} d\theta \right\}^4 \\
& \leq \left\{ (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta \right\}^2 \\
& \quad \times \left\{ (2\pi)^{-1} \int_0^{2\pi} f(\theta) |\varphi_n(d\mu, z)|^{-2} d\theta \right\}, \quad (4.13.15)
\end{aligned}$$

$z = \exp(i\theta)$. Letting $n \rightarrow \infty$, Theorem 4.13.2 yields

$$\begin{aligned}
& \left\{ (2\pi)^{-1} \int_0^{2\pi} (f(\theta) \mu'(\theta))^{1/4} d\theta \right\}^4 \\
& \leq \liminf_{n \rightarrow \infty} \left\{ (2\pi)^{-1} \int_0^{2\pi} |\varphi_n(d\mu, z)| (\mu'(\theta))^{1/2} d\theta \right\}^2 \\
& \quad \times \left\{ (2\pi)^{-1} \int_0^{2\pi} f(\theta) d\mu(\theta) \right\}, \quad (4.13.16)
\end{aligned}$$

$z = \exp(i\theta)$. Fix $\varepsilon > 0$, and choose a sequence $\{h_m\}$, $m = 1, 2, \dots$, of continuous 2π -periodic functions such that (4.11.3)–(4.11.5) hold, with μ substituted for ν . For $M = 1, 2, \dots$, let $f = f(\varepsilon, m, M)$ be defined by

$$f(\theta) = h_m \sigma_M((\varepsilon + \mu')^{-1}, \theta), \quad (4.13.17)$$

where σ_M denotes the arithmetic (i.e., Fejér) means of the trigonometric Fourier series of the functions under consideration. Applying (4.13.16) with this choice of f , and then first letting $m \rightarrow \infty$, then $M \rightarrow \infty$ and finally $\varepsilon \rightarrow 0$, we establish inequality (4.13.14), which, in turn, proves the theorem. ■

What I have described so far in this section is how the foundations of this new theory have started to be laid down. Due to the great variety of results and the extensive nature of their proofs, I have no hopes of providing the reader with an accurate portrayal of the present state of the art. Instead, I will state a few results which I expect to make the reader curious enough to turn to original sources such as [MáNeTo1, MáNeTo2, MáNeTo5–MáNeTo10, Ne19, Ne20, Ne24, Rah1, Rah4].

THEOREM 4.13.3 [MáNeTo9, 10]. *Let $\mu'_1 > 0$ almost everywhere, and let $d\mu_2 = g d\mu_1$, where the function $g > 0$ is such that $Rg \in L_\infty(d\mu_1)$ and $R/g \in L_\infty(d\mu_1)$ for some trigonometric polynomial R . Then*

$$\lim_{k \rightarrow \infty} \varphi_n(d\mu_2, z) \varphi_n(d\mu_1, z)^{-1} = \bar{D}(1/g, 1/z), \quad (4.13.18)$$

uniformly on every closed subset of the complement of the closed unit disk. If, in addition, at a real point t , the function g satisfies

$$g(t) > 0 \quad \text{and} \quad |g(t) - g(\theta)| \leq K |t - \theta| \quad (4.13.19)$$

for $|t - \theta| < \delta$ ($\delta > 0$ is fixed), then the asymptotic formula (4.13.18) also holds for $z = \exp(it)$.

THEOREM 4.13.4 [MáNeTo7]. *Let $d\alpha$ be such that $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha' > 0$ almost everywhere in $[-1, 1]$. Then, for every $f \in L_\infty$ and for every integer j , we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-1}^1 f(t) p_n(d\alpha, t) p_{n+j}(d\alpha, t) d\alpha(t) \\ = \pi^{-1} \int_{-1}^1 f(t) T_{|j|}(t) (1-t^2)^{-1/2} dt, \end{aligned} \quad (4.13.20)$$

where $T_{|j|}$ denotes the $|j|$ th Chebyshev polynomial of the first kind. Moreover, Turán's determinant $D_n(d\alpha)$ defined by

$$D_n(d\alpha, t) = p_n(d\alpha, t)^2 - p_{n+1}(d\alpha, t) p_{n-1}(d\alpha, t) \quad (4.13.21)$$

satisfies

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |D_n(d\alpha, t) \alpha'(t) - 2\pi^{-1}(1-t^2)^{1/2}| dt = 0. \quad (4.13.22)$$

The latter L_1 asymptotics for the Turán determinant D_n not only explains why D_n is nonnegative in all those special cases investigated by Turán [Tu1], Karlin and Szegő [KarSz], Askey [As1], and others (though it does not actually prove nonnegativity); it also has an invaluable application in finding absolutely continuous components of measures associated with orthogonal polynomials generated by three-term recurrences of the form (3.7). This program has been carried out consistently by Askey, Ismail, and their collaborators in a series of papers including [AsIs2, AsIs3, BanIs, BuIs, Is3-Is5, IsMu]. The point is that it is a matter of simple iteration to evaluate D_n in terms of the recurrence coefficients (3.8) when the orthogonal polynomials are defined recursively, and by my results proved in [Ne19, DoNe, MáNe3, MáNeTo4], one can show that Turán's determinant D_n converges pointwise under fairly weak conditions on the recurrence coefficients. Once we know that D_n converges, then of course, in view of (4.13.22), finding the limit poses no problems whatsoever.

The last result I mention in this section is one of my all-time favorites.

THEOREM 4.13.5 [MáNeTo5]. *Let $d\alpha$ be such that $\alpha' > 0$ almost everywhere in $[-1, 1]$ and, for every $\varepsilon > 1$, the set $\text{supp}(d\alpha) \setminus [-\varepsilon, \varepsilon]$ is finite. Then, for the corresponding Christoffel functions, the strong asymptotics*

$$\lim_{n \rightarrow \infty} \int_{-1}^1 |[n\lambda_n(d\alpha, t)]^{-1} \alpha'(t) - \pi^{-1}(1-t^2)^{-1/2}| dt = 0 \quad (4.13.23)$$

holds.

It is my sincere hope that the above selection of results regarding extensions of Szegő's theory will arouse the reader's appetite and stimulate his intellect to read more, learn more and contribute more to this subject.

4.14. Farewell to Orthogonal Polynomials in Finite Intervals

The purpose of this section is to assemble the pieces that are necessary to prove the estimate regarding the Lebesgue function $\Omega_n(d\alpha)$ which was formulated in Section 4.3 and which I claimed to be a simple application of the most significant results of the post-Szegő era of orthogonal polynomials cultivated by A. Máté, E. A. Rahmanov, V. Totik, and me. Recall that, for a given measure $d\alpha$, the Lebesgue function $\Omega_n(d\alpha)$ is defined by

$$\Omega_n(d\alpha, x) = \sup_{\|f\|_C \leq 1} |S_n(d\alpha, f, x)|, \quad (4.14.1)$$

where $C = C[-1, 1]$. Here $S_n(d\alpha, f)$ is the n th partial sum of the orthogonal Fourier series expansion of f in $p_n(d\alpha)$ (cf. (3.10) and (4.3.1)).

The result we have to prove here is Theorem 4.3.1. For the convenience of the reader I restate this theorem as

THEOREM 4.14.1 (Nevai). *Assume $\text{supp}(d\alpha) = [-1, 1]$ and $\alpha'(x) > 0$ almost everywhere in $[-1, 1]$. If α is continuous at $x \in [-1, 1]$, then*

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) \Omega_n(d\alpha, x)^2 = 0. \quad (4.14.2)$$

If α is uniformly continuous on a closed set $\mathcal{M} \subset (-1, 1)$, then (4.14.2) is satisfied uniformly for $x \in \mathcal{M}$. If, in addition, $\log \alpha'(\cos \theta) \in L_1$, then

$$\lim_{n \rightarrow \infty} n^{-1/2} \Omega_n(d\alpha, x) = 0 \quad (4.14.3)$$

almost everywhere in $[-1, 1]$. Finally, if α is continuous and positive on an interval $\Delta \subset [-1, 1]$, then (4.14.3) holds uniformly on every closed sub-interval of Δ .

Proof of Theorem 4.14.1. On the basis of (3.14), we can write

$$\Omega_n(d\alpha, x) = \int_{\mathbb{R}} |K_n(d\alpha, x, t)| d\alpha(t). \quad (4.14.4)$$

Fix $\varepsilon > 0$. Then

$$\Omega_n(d\alpha, x) = \int_{|x-t| < \varepsilon} |K_n(d\alpha, x, t)| d\alpha(t) + \int_{|x-t| \geq \varepsilon} |K_n(d\alpha, x, t)| d\alpha(t). \quad (4.14.5)$$

By Schwarz' inequality, we have

$$\begin{aligned} & \left[\int_{|x-t| < \varepsilon} |K_n(d\alpha, x, t)| d\alpha(t) \right]^2 \\ & \leq \int_{|x-t| < \varepsilon} d\alpha(t) \int_{|x-t| < \varepsilon} K_n(d\alpha, x, t)^2 d\alpha(t) \\ & \leq [\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] \int_{\mathbb{R}} K_n(d\alpha, x, t)^2 d\alpha(t) \\ & = [\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] \lambda_n(d\alpha, x)^{-1}. \end{aligned} \quad (4.14.6)$$

We use the Christoffel–Darboux formula (3.13) to estimate the second term on the right-hand side of (4.14.5). We obtain

$$\begin{aligned} & \int_{|x-t| \geq \varepsilon} |K_n(d\alpha, x, t)| d\alpha(t) \\ & \leq a_n \varepsilon^{-1} \int_{\mathbb{R}} |p_n(d\alpha, x)p_{n-1}(d\alpha, t) - p_{n-1}(d\alpha, x)p_n(d\alpha, t)| d\alpha(t) \\ & \leq a_n \varepsilon^{-1} \{ |p_n(d\alpha, x)| + |p_{n-1}(d\alpha, x)| \} \left[\int_{\mathbb{R}} d\alpha(t) \right]^{1/2}, \end{aligned} \quad (4.14.7)$$

where $a_n = a_n(d\alpha)$ is the recurrence coefficient in (3.7). Combining (4.14.5)–(4.14.7), we can conclude

$$\begin{aligned} & \lambda_n(d\alpha, x) \Omega_n(d\alpha, x)^2 \\ & \leq 2[\alpha(x+\varepsilon) - \alpha(x-\varepsilon)] + 4a_n^2 \varepsilon^{-2} \lambda^n(d\alpha, x) \{ p_n(d\alpha, x) \}^2 \\ & \quad + p_{n-1}(d\alpha, x)^2 \int_{\mathbb{R}} d\alpha(t). \end{aligned} \quad (4.14.8)$$

Now if $\alpha' > 0$ a.e. in $[-1, 1]$, then by Rahmanov's Theorem 4.5.7,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \quad (4.14.9)$$

(cf. (4.5.11)), and by Theorem 4.5.6,

$$\lim_{n \rightarrow \infty} \lambda_n(d\alpha, x) p_n(d\alpha, x)^2 = 0 \quad (4.14.10)$$

(cf. (4.5.10)) for every $x \in [-1, 1]$. Moreover, (4.14.10) holds uniformly in every closed subinterval of $(-1, 1)$. Therefore, by (4.14.8)–(4.14.10), the asymptotics (4.14.2) is satisfied at every point of continuity of α . If we also assume that $\log \alpha'(\cos \theta) \in L_1$, then, by Theorem 4.5.1, we have

$$\limsup_{n \rightarrow \infty} [n\lambda_n(d\alpha, x)]^{-1} < \infty \quad (4.14.11)$$

(cf. (4.5.1)) for almost every x in $[-1, 1]$, and thus (4.14.3) follows from (4.14.2) and (4.14.11). Finally, the statement regarding uniform convergence in (4.14.3) is a consequence of (4.14.2) and Theorem 4.5.2, where uniform estimates are given for (4.14.11). ■

In the hope that I have succeeded in fulfilling my elaborate plan to take the reader on an exciting journey through some aspects of the general theory of polynomials, orthogonal on bounded intervals, I now set out to expand our horizon by moving on to the second major topic of this study which consists of polynomials orthogonal on the whole real line.

PART 2: ORTHOGONAL POLYNOMIALS ON INFINITE INTERVALS

In this Part, all measures will be absolutely continuous, say $d\alpha = w dx$, and we use the notation $p_n(w, x)$, $\lambda_n(w, x)$, etc., instead of $p_n(d\alpha, x)$, $\lambda_n(d\alpha, x)$, and so forth. The function w is referred to as a weight function.

4.15. *Freud Weights*

Freud's contributions to the theory of polynomials orthogonal on bounded intervals are by no means as significant as those of Szegő, who almost single-handedly laid down the foundations of a powerful theory when the associated measure is supported on a compact interval and the absolutely continuous component of the measure satisfies Szegő's condition of logarithmic integrability. Sometimes I wonder what would have happened if Szegő had tried to apply his unsurpassable ingenuity and analytic skills to orthogonal polynomials on infinite intervals. It baffles me why Szegő did not attempt to create a general theory of orthogonal polynomials on infinite intervals. I have no doubt that had he initiated research in this direction earlier, say, half a century ago, by now we would have an essen-

tially completed theory of orthogonal polynomials associated with measures with noncompact support.

For Freud this presented a wonderful and practically unmissable opportunity to carve his name in the history book on orthogonal polynomials as the founder of a new theory. As a result of Freud's juggernaut energy, in the last 10 years of his life he introduced a class of polynomials which we now call the Freud polynomials. They are the subject of the remaining sections of this work.

Twenty years ago there was a great amount of information available regarding some orthogonal polynomials on infinite intervals for which one could find characterizations in terms of explicit special functions, differential equations, generating functions, recursive formulas, and so forth. As examples I mention the Hermite, generalized Hermite, Laguerre, Lommel, Meixner, Poisson–Charlier, Pollaczek, and Stieltjes–Wiegert polynomials (cf. [As6, Chi3, Sz2]). While working on problems related to the uniqueness of the solution of the moment problem, on convergence of Gauss–Jacobi quadrature, orthogonal Fourier series and Lagrange interpolation (cf. [Fr19–Fr21, Fr23, Fr24, Fr26, Fr32, Fr33]) and on his book [Fr31a, b] in the sixties, Freud realized not only that there had been a complete lack of results regarding general orthogonal polynomials on infinite intervals but also that the then available tools of the trade did not enable one to obtain such results without undue efforts of mostly an ad hoc nature. Moreover, being an approximator of considerable breadth, Freud also set his eye on extending and expanding the Jackson–Bernstein–Timan's theory of direct and converse theorems of approximation theory to infinite intervals. It was this goal which directed Freud towards general orthogonal polynomials on infinite intervals. He reasoned as follows. If one wishes to approximate, then one has to be able to construct tools for such an approximation; although best approximation might be difficult if not impossible to achieve by simple means, one should be able to produce nearly best approximations; the way to a man's best approximation is via delayed arithmetic (i.e., de la Vallée–Poussin) means of orthogonal Fourier series; behind every bounded delayed arithmetic mean there is a nearly positive $(C, 1)$ mean; $(C, 1)$ means and Christoffel functions live and thrive together; there are no Christoffel functions without orthogonal polynomials. The above line of reasoning is more than just a pure guess on my part, as to the nature of Freud's reflections. As a matter of fact, I could have used direct quotation marks (allowing a certain poetic freedom) since I was privileged to have conducted long conversations with him regarding the way he arrived at the conclusion that it was time to move the emphasis to the whole real line and to orthogonal polynomials there.

Quite understandably, Freud took the Hermite polynomials h_n as the

cornerstone and prime example of orthogonal polynomials with weights whose support is noncompact (they are related to Fourier transforms, one of the basic concepts in harmonic analysis). The Hermite polynomials are orthogonal with respect to $\exp(-x^2)$. Thus the proper generalization would be considering orthogonal polynomials associated with either $\exp(-x^{2m})$, m a natural number, or $\exp(-|x|^m)$, $m > 0$, or $\exp(-Q(x))$, Q being of a prescribed growth. These weight functions and their slight variations are the ones which various authors these days are inclined to call Freud weights.

Freud's first paper on this subject is [Fr20], where he considers weight functions w which satisfy

$$C_1 \exp(-Cx^2) \leq w(x) \leq C_2 \exp(-Cx^2), \quad x \in \mathbb{R}, \quad (4.15.1)$$

where C , C_1 and C_2 are positive constants. Using ideas discussed and dissected in Section 4.4., Freud proves that the orthogonal Fourier series associated with $p_n(w)$ is $|C, 1|$ summable almost everywhere on the real line for all $f \in L_2(w)$. Naturally, as the reader is expected to anticipate at this point, it is the Christoffel function of the Hermite polynomials which plays the role of the drum major.

In [Fr32] Freud takes a deep dive and introduces the Q 's and q_n 's we (the experts) are all familiar with. What I refer to is weights w of the form

$$w(x) = \exp(-Q(x)), \quad x \in \mathbb{R}, \quad (4.15.2)$$

where $Q > 0$ is an even C^1 function on \mathbb{R} such that $xQ'(x)$ increases for $x > 0$ and $Q'(x) \rightarrow \infty$ as $x \rightarrow \infty$. For such a function Q , the numbers $q_n = q_n(Q)$ are the unique positive solutions of the equation $xQ'(x) = n$, $n = 1, 2, \dots$. Let me point out that these Q 's and q_n 's were actually introduced by M. M. Dzrabasyan and A. B. Tavadyan (cf. [Dz, DzTa]), who used them to characterize the rate of weighted best polynomial approximations of functions of several variables. (H. N. Mhaskar and E. B. Saff's [MhSa5, formula (3.7), p. 77] should also be mentioned where a quantity, a_n , of the same order of magnitude as q_n is defined as a solution of a certain equation. This a_n is expected to play an important role in the theory of weighted polynomial approximation.) What Freud does in [Fr32] is to generalize results of M. M. Dzrabasyan and A. B. Tavadyan for approximation on the real line in the one-variable case. Fortunately, Freud did not stop here, and for the next (and last) 10 years of his life his research mostly revolved around problems associated with the weight (4.15.2) in both orthogonal polynomials and approximation theory.

With all due respect to Freud, I must point out that he was completely unaware of two papers of J. Shohat, [Sho3, Sho7], where weight functions of the form $\exp(-\Pi(x))$, $x \in \mathbb{R}$, are introduced and the corresponding

orthogonal polynomials are shown to satisfy second-order linear differential equations with variable coefficients. I first heard of these papers in 1982 from R. Askey; earlier they seem to have been resting in oblivion.

At the present time the theory of orthogonal polynomials with Freud-type weight functions has reached a state far beyond infancy. This is in sharp contrast to my characterization of this theory in my paper [Ne29] in 1982, where I declared it to be virtually nonexistent. The past 4 years have produced a number of extraordinary events which have started the mature development of this subject. As a matter of fact, Freud himself never expected such fast progress, and he would certainly be most surprised to learn about the latest developments concerning his polynomials. As the reader will soon see, while Freud initiated the investigation of most problems in orthogonal polynomials with Freud-type weights, his results have since been surpassed in almost every respect in both sharpness and generality. The responsibility (or, rather, honor) for improving and/or outdating Freud's results is to be shared by W. C. Bauldry, S. S. Bonan, A. L. Levin, D. S. Lubinsky, Al. Magnus, A. Máté, H. Mhaskar, E. A. Rahmanov, E. B. Saff, R. C. Sheen, V. Totik, J. L. Ullman, and me.

4.16. Christoffel Functions for Freud Weights

Freud started by estimating Christoffel functions for Hermite weights in [Fr20] (lower bounds) and [Fr33] (upper bounds) in 1963 and 1968, respectively. In the former, Freud applied a rather ad hoc approach based on Mehler's formula

$$\sum_{k=0}^{\infty} p_k(w, x)^2 t^k = \pi^{-1/2} (1-t^2)^{-1/2} \exp\{2tx^2/(1+t)\}, \quad (4.16.1)$$

where $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$, is the Hermite weight function (cf. [Sz2, p. 102]). What Freud noticed was that, putting $t = 1 - 1/n$ in (4.16.1) and making some elementary estimates, one can easily conclude

$$w(x) \lambda_n(w, x)^{-1} \leq \text{const} \cdot n^{1/2}, \quad x \in \mathbb{R}, \quad (4.16.2)$$

where the constant is independent of n and x . This is in sharp contrast with the estimate

$$\max_{x \in \mathbb{R}} [w(x) p_n(w, x)^2] \sim n^{-1/6} \quad (4.16.3)$$

(cf. [Sz2, p. 242]). However, it is also well known that, for every $0 < \varepsilon < 1$,

$$\max_{|x| \leq \varepsilon(2n)^{1/2}} [w(x) p_n(w, x)^2] \sim n^{-1/2} \quad (4.16.4)$$

(cf. [Sz2, p. 242]), and thus (4.16.2) amounts to a $(C, 1)$ extension of (4.16.4) to the whole real line. Since generating functions such as (4.16.1) exist only for a small privileged class of orthogonal polynomials, it is clear that one should not expect to be able to apply this method for more than a handful of weight functions.

The upper bound for the Christoffel functions of Hermite weights w was found in [Fr33] by an equally ad hoc method; namely, first Freud used Sturm's comparison theorem (cf. [Sz2, pp. 19–21]) to find upper bounds for the distances between consecutive zeros of Hermite polynomials, and then he applied the Markov–Stieltjes inequality (cf. [Fr31b, p. 29]) to obtain upper bounds for $\lambda_n(w, x)$. In this way he proved

$$w(x)^{-1} \lambda_n(w, x) \leq \text{const} \cdot n^{-1/2}, \quad |x| \leq \varepsilon(2n)^{1/2}, \quad (4.16.5)$$

for every fixed $0 < \varepsilon < 1$. Here, again, we face the same obstacle as before; namely, there are no convenient differential equations available for general weight functions for which one can find a comparison system whose solutions have zeros with known behavior.

Naturally, in view of the extremal property (4.1.1), all estimates involving Christoffel functions of Hermite weights will result in similar estimates for all weights w whose size is comparable to $\exp(-x^2)$.

It took several years for Freud to realize that essentially all barriers associated with infinite intervals can be removed by a clever argument which enables one to estimate weighted L_p norms of polynomials in terms of integrals over finite intervals. The first such infinite–finite range inequality (an expression coined by D. S. Lubinsky) was proved by Freud in [Fr36] (in L_∞) and [Fr40, Lemma 1, p. 570] (in $L_2(w)$) for Hermite weights and in [Fr50, Theorem 2, p. 127] for a wider class of weights. It can be formulated as follows.

THEOREM 4.16.1 [Fr50]. *Let w be defined by*

$$w(x) = \exp(-x^m), \quad x \in \mathbb{R}, \quad (4.16.6)$$

where m is an even positive integer. Then there exists a positive number c such that, for every $n = 1, 2, \dots$, the inequality

$$\int_{\mathbb{R}} \Pi(t)^2 w(t) dt \leq 2 \int_{-cn^{1/m}}^{cn^{1/m}} \Pi(t)^2 w(t) dt \quad (4.16.7)$$

holds for all polynomials Π of degree at most n .

This inequality, in my global evaluation of Freud's contributions to orthogonal polynomials, gets a very high rating indeed. It turned out to be the basis of a whole new theory of orthogonal polynomials associated with

Freud-type weight functions. The infinite-finite range inequality (4.16.7) combined with the extremal property (4.1.1) immediately yields

$$\lambda_n(w, x) \sim \lambda_n(w_{(n)}, x), \quad (4.16.8)$$

uniformly for all real x and $n = 1, 2, \dots$, where $w_{(n)}(x) = w(x)$ for $|x| \leq cn^{1/m}$ and $w_{(n)}(x) = 0$ otherwise. Hence estimating Christoffel functions of weights with unbounded support is reduced to estimating Christoffel functions of variable weights with compact support. Thus all the machinery of Christoffel functions on finite intervals can be brought in to investigate the case of weight functions on infinite intervals.

Although Freud's original proof of (4.16.7) was exceedingly complicated, it was subsequently simplified by several authors. In [Fr50] it was necessary for Freud to find suitable one-sided approximations for the weight function w in (4.16.6) when proving (4.16.7), and thus the assumption that m in (4.16.6) is an even positive integer could not be relaxed. It did not take long for me to realize that, in fact, one could avoid using one-sided approximations via a straightforward proof (cf. [Ne11, Lemma 3.2, p. 339]) which leads to significant generalizations of Freud's infinite-finite range inequality. On the basis of my results in [Ne11, Ne19], one can easily prove the following, which is closely related to a theorem of W. C. Bauldry [Bau1].

THEOREM 4.16.2 (Nevai). *Let w be defined by*

$$w(x) = |x|^a \exp(-|x|^m), \quad x \in \mathbb{R}, \quad (4.16.9)$$

where $a > -1$ and $m > 0$. Let $p > 0$ and $b \in \mathbb{R}$ be given. Then there exist positive numbers c and d such that, for every $n = 1, 2, \dots$, the inequality

$$\int_{|t| \geq cn^{1/m}} |\Pi(t)|^p |t|^b w(t) dt \leq \exp(-dn) \int_{-n^{1/m}}^{cn^{1/m}} |\Pi(t)|^p w(t) dt \quad (4.16.10)$$

holds for all polynomials Π of degree at most n . In particular, we have

$$\int_{\mathbb{R}} |\Pi(t)|^p w(t) dt \leq 2 \int_{-cn^{1/m}}^{cn^{1/m}} |\Pi(t)|^p w(t) dt. \quad (4.16.11)$$

Proof of Theorem 4.16.2. Let Π be a polynomial of degree at most n . In what follows, K will denote positive constants independent of n and x . According to my results on generalized Christoffel functions (Ne19, Theorem 6.3.28, p. 120], we have

$$|\Pi(x)|^p \leq Kn^{a+3} \int_{-1}^1 |\Pi(t)|^p |t|^a dt, \quad (4.16.12)$$

for $|x| \leq 1$ (cf. Theorem 4.7.6). Thus by an inequality of S. N. Bernstein [Be5, p. 21],

$$|\Pi(x)|^p \leq K(2|x|)^{2n} n^{a+3} \int_{-1}^1 |\Pi(t)|^p |t|^a dt, \quad |x| \geq 1, \quad (4.16.13)$$

and applying (4.6.13) with $\Pi(n^{1/m}t)$, we obtain

$$|\Pi(x)|^p \leq \exp(Kn) n^{-2n/m} |x|^{2n} \int_{-n^{1/m}}^{n^{1/m}} |\Pi(t)|^p |t|^a dt, \quad |x| \geq n^{1/m}, \quad (4.16.14)$$

from which

$$|\Pi(x)|^p \leq \exp((K+1)n) n^{-2n/m} |x|^{2n} \int_{-n^{1/m}}^{n^{1/m}} |\Pi(t)|^p w(t) dt, \quad |x| \geq n^{1/m}, \quad (4.16.15)$$

follows. Hence

$$\begin{aligned} & \int_{|x| \geq cn^{1/m}} |\Pi(x)|^p |x|^b w(x) dx \\ & \leq \exp((K+1)n) n^{-2n/m} \int_{cn^{1/m}}^{\infty} |x|^{2n+b} w(x) dx \int_{-n^{1/m}}^{n^{1/m}} |\Pi(t)|^p w(t) dt, \end{aligned} \quad (4.16.16)$$

and now the infinite-finite range inequality (4.16.10) is a consequence of asymptotic formulas on incomplete gamma functions (cf. [BatEr, Chap. 9]) which guarantee the existence of c such that

$$\exp((K+1)n) n^{-2n/m} \int_{cn^{1/m}}^{\infty} |x|^{2n+b} w(x) dx \leq \exp(-dn) \quad (4.16.17)$$

for all $n \in \mathbb{N}$. ■

After the initial papers [Fr40, Fr50, FrNe2, Ne11, Ne9], Freud produced a large number of publications (cf. [Fr36, Fr44, Fr48, Fr49, Fr51–Fr54, Fr58, Fr59, Fr69, FrGiRa2]) improving his proof of the infinite-finite range inequality (4.16.7) and leading to extensive generalizations for Freud weights given by

$$w(x) = \exp(-Q(x)), \quad x \in \mathbb{R}. \quad (4.16.18)$$

His work was continued and complemented by W. C. Bauldry, S. S. Bonan, A. L. Levin, D. S. Lubinsky, H. N. Mhaskar, E. B. Saff, and R. S. Varga (cf. [Bau1, Bau2, Bon1, MhSa1–MhSa5, LevLu1, LevLu2, Lu2–Lu4, Lu6,

SaVa1, SaVa2]). In all fairness, it must be pointed out that some of this work was independent of Freud's research, such as the investigations of E. B. Saff and R. S. Varga regarding weighted L_∞ norms of polynomials. A typical result is the following

THEOREM 4.16.3 [Lu2]. *Let w be defined by (4.6.18), where Q is even and continuous in \mathbb{R} , and assume that there exists $A > 0$ such that $Q'(x)$ exists and $xQ'(x)$ is increasing in $[A, \infty)$. Let q_n be the unique positive root of the equation $qQ'(q) = n$, for n sufficiently large. Then, for every $0 < p \leq \infty$, there exist positive constants n_0 and c , depending on w and p only, such that, for every $n \geq n_0$,*

$$\left[\int_{\mathbb{R}} |\Pi(t) w(t)|^p dt \right]^{1/p} \leq c \left[\int_{-11q_{2n}}^{11q_{2n}} |\Pi(t) w(t)|^p dt \right]^{1/p} \tag{4.16.19}$$

for all polynomials Π of degree at most n .

Applying (4.16.19) with $p = 2$ and using the extremal property (4.1.1), we again obtain (4.16.8), where $w_{(n)}(x) \sim w(x)$ for $|x| \leq cq_n$ and $w_{(n)}(x) = 0$ otherwise.

The next step towards estimating Christoffel functions of Freud weights consists of approximating these weights and their reciprocals by polynomials on sufficiently large intervals. For instance, for the Hermite weight $w(x) = \exp(-x^2)$, one can easily construct two polynomials P and R of degree at most n such that

$$P(x) \leq \exp(x^2), \quad x \in \mathbb{R}, \tag{4.16.20}$$

$$P(x) \geq \text{const} \cdot \exp(x^2), \quad |x| \leq Kn^{1/2}, \tag{4.16.21}$$

and

$$R(x) \sim \exp(-x^2), \quad |x| \leq Kn^{1/2}, \tag{4.16.22}$$

with some suitable positive constants. This can be achieved by choosing P and R to be the n th partial sums of the Taylor expansion of $\exp(x^2)$ and $\exp(-x^2)$, respectively. Since all the Taylor coefficients of $\exp(x^2)$ are positive, inequality (4.16.20) follows immediately, whereas (4.16.21) and (4.16.22) can be proved by examining the remainder terms of the Taylor series. The same argument works for $w(x) = \exp(-x^m)$, $m > 0$ even, as well (cf. [Fr50]). However, for the weight $w(x) = \exp(-|x|^m)$, $m > 1$, or for the more general $w(x) = \exp(-Q(x))$, taking partial sums of power series does not seem to be reasonable, since these weights are no longer entire functions. As Freud noticed (cf. [Fr51]), one can circumvent the problem caused by the lack of analyticity by first approximating Q by a polynomial

P^* and then taking partial sums of the Taylor expansion of $\exp(P^*)$ and $\exp(-P^*)$, respectively. Freud's next observation regarding inequalities of the form (4.16.20)–(4.16.22) was that it is not really necessary to prove two-sided approximations for all values of x under consideration, when estimating Christoffel functions. Instead, it is sufficient to find a polynomial $P = P_t$ for all t with $|t| \leq Kq_n$ such that $P(t) = 1/w(t)$ and

$$|P(x)| \leq \text{const}/w(x), \quad |x| \leq K_1 q_n \quad (4.16.23)$$

(cf. [Fr58, Lemma 3.2, p. 291; Fr54, Lemma 3.2, p. 161]).

D. S. Lubinsky's arrival at the scene a few years ago has completely changed our perceptions of the possibilities of approximating Freud-type weights by polynomials. Lubinsky was convinced that, while Freud weights might not be analytic, they still should be approximable by entire functions whose Taylor sums could be kept under control. According to T. Carleman's theorem in [Ca2], if f and $g > 0$ are continuous in \mathbb{R} , then there exists an entire function G such that

$$|f(x) - G(x)| \leq g(x), \quad x \in \mathbb{R}. \quad (4.16.24)$$

Hence, if $w = \exp(-(Q(x)))$, $x \in \mathbb{R}$, where Q is continuous, then, for every $\varepsilon > 0$, there are two entire functions G_1 and G_2 such that

$$1 - \varepsilon < w(x)/G_1(x) < 1 + \varepsilon, \quad x \in \mathbb{R}, \quad (4.16.25)$$

and

$$1 - \varepsilon < w(x)^{-1}/G_2(x) < 1 + \varepsilon, \quad x \in \mathbb{R}. \quad (4.16.26)$$

Unfortunately, one cannot control the behavior of the Taylor coefficients of G_1 and G_2 above. Lubinsky [Lu3] came up with the idea of considering G defined by

$$G(x) = 1 + \sum_{n=1}^{\infty} (x/q_n)^{2n} n^{-1/2} w(q_n)^{-1} \quad (4.16.27)$$

for $w(x) = \exp(-(Q(x)))$ with $q_n Q'(q_n) = n$. This construction turned out to be the appropriate one for Freud weights, as shown by

THEOREM 4.16.4 [Lu3]. *Let w be defined by*

$$w(x) = \exp(-Q(x)), \quad x \in \mathbb{R}, \quad (4.16.28)$$

where Q is even and continuous in \mathbb{R} , and assume that there exist $A > 0$, $B > 0$ and $0 < \theta < 1$ such that Q'' exists in $[A, \infty)$, Q' is positive in $[A, \infty)$ and the inequality $-\theta \leq xQ''(x)/Q'(x) \leq B$ holds for every $x \geq A$. Let q_n be the unique positive root of the equation $qQ'(q) = n$ for n sufficiently large.

Then G , defined by (4.16.27) is an even entire function satisfying

$$c_1 \leq G(x) w(x) \leq c_2, \quad x \in \mathbb{R}, \quad (4.16.29)$$

where c_1 and c_2 are positive constants.

Infinite-finite range inequalities and weight approximations such as (4.16.19) and (4.16.29) enable one to obtain upper bounds for Christoffel functions associated with Freud weights via Christoffel function estimates for weights with compact support. In order not to confuse the reader with too many conditions on the weight function and to let the ideas shine through, I will only state and prove the following theorem of Freud [Fr58, Theorem 3.1, p. 292].

THEOREM 4.16.5 [Fr58]. *Let w be defined by*

$$w(x) = \exp(-2Q(x)), \quad x \in \mathbb{R}, \quad (4.16.30)$$

where Q is even and convex in \mathbb{R} . Assume that $Q'(x) > 0$ for $x > 0$, and there are three constants: $a > 1$, $b > 0$ and $c > 0$ such that

$$a < Q'(2x)/Q'(x) \quad \text{and} \quad xQ''(x)/Q'(x) < b, \quad x \geq c. \quad (4.16.31)$$

Then there exist two positive constants, A and B , such that

$$\lambda_n(w, x)/w(x) \leq Aq_n/n, \quad |x| \leq Bq_n, \quad (4.16.32)$$

where q_n is the positive root of the equation $qQ'(q) = n$.

Proof of Theorem 4.16.5. We use the symbol K to denote positive constants independent of all variables. Let P denote the $[n/2]$ th partial sum of the Taylor series of G in (4.16.27). Then obviously

$$P(x)^2 w(x) \leq c_2, \quad x \in \mathbb{R}. \quad (4.16.33)$$

Moreover, examination of the remainder term (cf. [Lu3]) shows that there is a constant B such that

$$P(x)^2 w(x) \geq \text{const}, \quad |x| \leq Bq_n. \quad (4.16.34)$$

Applying (4.16.19) (with $p = 2$) and (4.1.1), we obtain

$$\lambda_n(w, x) \leq K \min_{\substack{\Pi \in \mathbb{P}_{[n/2]} \\ \Pi(x) = 1}} \int_{-Kq_n}^{Kq_n} |\Pi(t)|^2 [P(t)/P(x)]^2 w(t) dt. \quad (4.16.35)$$

Thus, by (4.16.33) and (4.16.34),

$$\lambda_n(w, x)/w(x) \leq K \min_{\substack{\pi \in \mathbb{P}_{[n/2]} \\ \pi(x) = 1}} \int_{-Kq_n}^{Kq_n} |\Pi(t)|^2 dt \quad (4.16.36)$$

for $|x| \leq Bq_n$. By a change of variable, $t' = t/Kq_n$, the minimum on the right-hand side of (4.16.36) becomes the Christoffel function $\lambda_{[n/2]+1}(dL, x/(Kq_n))$ of the Lebesgue dL measure in $[-1, 1]$. More precisely, we obtain

$$\lambda_n(w, x)/w(x) \leq Kq_n \lambda_{[n/2]+1}(dL, x/(Kq_n)), \quad |x| \leq Bq_n. \quad (4.16.37)$$

Since

$$\lambda_n(dL, x) \leq K/n, \quad x \in \mathbb{R} \quad (4.16.38)$$

(cf. [Fr31b, p. 103]), the theorem follows from (4.16.37). ■

In view of (4.1.1), one is led to believe that lower estimates of Christoffel functions do not need the application of infinite–finite range inequalities. This is indeed the case as long as we are interested in estimates on intervals such as $[-Bq_n, Bq_n]$. It turns out, however, that it is possible to find lower bounds for Christoffel functions that are valid on the whole real line, but proving such estimates does require application of infinite–finite range inequalities. Another difficulty enters the picture when one is looking for lower bounds. Namely, while it is relatively easy to approximate $1/w$ by entire functions with positive Taylor coefficients, it is much more strenuous to do the same for w , and thus polynomial approximations to w are more delicate in nature than those to $1/w$. It is exactly the latter approximations which enable one to find lower bounds for Christoffel functions for Freud weights.

For w given by (4.16.30), it is hopeless to search for polynomials P such that $P^2 \leq w$ on the whole real line. Nevertheless, under various conditions on Q , it is possible to show the existence of $B > 0$ such that, for every n , there is a polynomial P of degree at most n satisfying

$$P(x)^2 \sim w(x), \quad |x| \leq Bq_n. \quad (4.16.39)$$

The construction of such polynomials has been discussed in several papers by Freud (cf. [Fr54, Fr58]), and Freud's results were later significantly improved by A. L. Levin and D. S. Lubinsky in [LevLu1, LevLu2].

THEOREM 4.16.6 [Fr54, LevLu2]. *Let $w = \exp(-2Q)$, where Q is even and continuous in \mathbb{R} . Assume that there exist $a > 0$, $b > 0$ and $0 < \theta < 1$ such*

that Q'' is continuous in $[a, \infty)$, Q' is positive in $[a, \infty)$ and $-\theta \leq tQ''(t)/Q'(t) \leq b$ for $t \in [a, \infty)$ while, for some $d > 1$ ($d \neq 2, 4$) $tQ''(t)/Q'(t) \rightarrow d$ as $t \rightarrow \infty$. Then there exists a positive constant A such that

$$\lambda_n(w, x)/w(x) \geq Aq_n/n \quad (4.16.40)$$

for every $x \in \mathbb{R}$.

Proof of Theorem 4.16.6. This proof consists of three parts. The symbol K is used to denote appropriate positive constants.

Step 1. Here we show (4.16.40) for $|x| \leq Bq_n$, where B is a suitable constant. Let P be a polynomial of degree at most n such that (4.16.39) holds (cf. [LevLu2]). Then by (4.1.1)

$$\begin{aligned} \lambda_n(w, x) &\geq K \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(x) = 1}} \int_{-Bq_n}^{Bq_n} |\Pi(t)|^2 w(t) dt \\ &\geq Kw(x) \min_{\substack{\Pi \in \mathbb{P}_{n-1} \\ \Pi(x) = 1}} \int_{-Bq_n}^{Bq_n} |\Pi(t)|^2 |P(t)/P(x)|^2 dt \\ &\geq Kw(x) \min_{\substack{\Pi \in \mathbb{P}_{2n-1} \\ \Pi(x) = 1}} \int_{-Bq_n}^{Bq_n} |\Pi(t)|^2 dt, \end{aligned} \quad (4.16.41)$$

and thus

$$\lambda_n(w, x)/w(x) \geq Aq_n \lambda_{2n}(dL, x/(Bq_n)) \quad (4.16.42)$$

for $|x| \leq Bq_n$, there again dL denotes the Lebesgue measure in $[-1, 1]$. Now the lower estimate of the Christoffel functions of Legendre polynomials (cf. [Fr31b, p. 104]) yields (4.16.40) for $|x| \leq Bq_n/2$.

Step 2. Now we prove (4.16.40) for $|x| \leq Cq_n$, where C is an arbitrary constant. First we show that

$$\lim_{m \rightarrow \infty} q_n/q_{mn} = 0, \quad (4.16.43)$$

uniformly in $n = 1, 2, \dots$. We have

$$\begin{aligned} \log(q_n/q_{mn}) &= \log(1/m) - \log Q'(q_n) + \log Q'(q_{mn}) \\ &= \log(1/m) + \int_{q_n}^{q_{mn}} Q''(t)/Q'(t) dt. \end{aligned} \quad (4.16.44)$$

Therefore by (4.16.31),

$$\log(q_n/q_{mn}) \leq \log(1/m) + b \int_{q_n}^{q_{mn}} 1/t \, dt = \log(1/m) - b \log(q_n/q_{mn}), \quad (4.16.45)$$

that is,

$$\log(q_n/q_{mn}) \leq (1+b)^{-1} \log(1/m), \quad (4.16.46)$$

and letting $m \rightarrow \infty$, (4.16.43) follows. Now we can prove (4.16.40), for $|x| \leq Cq_n$, as follows. Since $\lambda_n(w)$ is a decreasing function of n , we have by Step 1,

$$\lambda_n(w, x)/w(x) \geq \lambda_{mn}(w, x)/w(x) \geq Aq_{mn}/(nm) \quad (4.16.47)$$

for $|x| \leq Bq_{mn}$ with some $B > 0$. By (4.16.43) for given $B > 0$ and $C > 0$, there exists m such that $Cq_n \leq Bq_{mn}$. Moreover, $q_{mn} > q_n$ because $xQ'(x)$ increases. Consequently, (4.16.40) does indeed hold for $|x| \leq Cq_n$.

Step 3. Finally, we note that, for $|x| \geq Dq_n$ with sufficiently large D , inequality (4.16.40) follows immediately from Step 2 and Theorem 4.16.3 applied with $p = \infty$. ■

*4.17. Orthogonal Fourier Series, Cesàro
and de la Vallée–Poussin Means, and Bernstein–Markov
and Nikolskii Inequalities with Freud Weights*

There are short reasons for this long title: if I devoted an individual section to each of the topics mentioned in the title, then the length of this paper would exceed the upper limit of what I would expect from the reader in terms of undivided attention, curiosity, good will, and patience. Besides, these subjects, though exceptionally appealing in their own right, were mostly developed by Freud in connection with his research on weighted polynomial approximations (cf. [DiTo1, DiTo2, DiLuNeTo, Mh4, MhSa2, NeTo1, NeTo2, Sa]) where they were used as auxiliary tools of the trade rather than primary subjects of investigation. My intention of limited discussion is also fueled by my conviction that new results are being obtained and old ones are being improved at such a pace that every attempt to provide a reasonably thorough description of the subject is liable to fail anyway.

The name of the game is characterization of the rate of weighted mean approximation of functions by polynomials in terms of suitable moduli of smoothness; the rules of the game were mostly instituted by S. N. Bernstein; the tools of the game are orthogonal Fourier series, Cesàro and

de la Vallée–Poussin means, Bernstein–Markov and Nikolskii-type inequalities with Freud weights, and Christoffel functions; whereas the principal players are Freud, Z. Ditzian, D. S. Lubinsky, H. N. Mhaskar, E. B. Saff, V. Totik, and Yours Truly.

In what follows, w is a Freud weight defined by

$$w(x) = \exp(-Q(x)), \quad x \in \mathbb{R}, \quad (4.17.1)$$

where Q satisfies some conditions specified later which essentially guarantee that $Q(x)$ behaves similarly to $|x|^m$ with some $m > 1$.

The first objective is to construct a proper means of approximation which is almost as good as the best approximating polynomial. Having had extensive experience with Cesàro sums of orthogonal Fourier series (cf. Section 4.4), Freud chose to select these sums as building blocks.

Let f be a function on the real line such that $f^2 w \in L_\infty$. Let $S_n(w, f)$ denote the n th partial sum of the orthogonal Fourier series expansion of f in $\{p_k(w)\}$ (cf. (3.10)). Let q_n be the solution of $qQ'(q) = n$. The conditions imposed on Q will always ensure that q_n is uniquely determined for large enough values of n . For a given x , let I_n and E_n be defined by

$$I_n = (x - q_{2n}/n, x + q_{2n}/n) \quad \text{and} \quad E_n = \mathbb{R} \setminus I_n. \quad (4.17.2)$$

Then, as in Section 4.4, we can write

$$\begin{aligned} S_k(w, f, x) &= \int_{\mathbb{R}} f(t) K_k(w, x, t) w(t) dt \\ &= \int_{I_n} f(t) K_k(w, x, t) w(t) dt + \int_{E_n} f(t) K_k(w, x, t) w(t) dt, \end{aligned} \quad (4.17.3)$$

and both terms on the right-hand side of (4.17.3) can be estimated in exactly the same way as it was done in (4.4.4)–(4.4.11). The only difference is that, this time, estimates of Christoffel functions discussed and described in Section 4.16 are used and the bound (4.4.8) for the recursion coefficient a_k is replaced by

$$a_k = \gamma_{k-1}/\gamma_k = \int_{\mathbb{R}} t p_{k-1}(w, t) p_k(w, t) d\alpha(t) \leq \text{const } q_{2n}, \quad (4.17.4)$$

which follows immediately from infinite–finite range inequalities such as Theorem 4.15.3 under suitable conditions on Q .

This argument yields L_∞ boundedness of Cesàro sums which can be extended to L_1 boundedness by a standard duality argument. In between L_1 and L_∞ one can apply M. Riesz and G. O. Thorin's interpolation theorem (cf. [Zy2, p. 93]). This is how Freud proved the boundedness of

Cesàro means of orthogonal Fourier series in a number of papers, under various conditions on w (cf. [Fr40, Fr41, Fr43, Fr47, Fr48, Fr51–Fr54, Fr57, Fr62, Fr66, Fr69]). The following is a representative result taken from [Fr54, Theorem 4.2, p. 166].

THEOREM 4.17.1 [Fr54]. *Let w be given by (4.17.1), where*

$$0 < Q''(t) \leq (1 + c_1) Q''(x), \quad c < t < x, \quad (4.17.5)$$

$$Q''(2t) \geq (1 + c_2) Q''(t), \quad t > c, \quad (4.17.6)$$

and

$$tQ''(t)/Q'(t) \leq c_3, \quad t > c, \quad (4.17.7)$$

with some suitable positive constants c , c_1 , c_2 and c_3 . Then, for every $1 \leq p \leq \infty$, there exists a constant K independent of n such that

$$\left\{ \int_{\mathbb{R}} |n^{-1} \sum_{k=1}^n S_k(w, f, x) w(t)^{1/2}|^p dt \right\}^{1/p} \leq K \left\{ \int_{\mathbb{R}} |f(t) w(t)^{1/2}|^p dt \right\}^{1/p} \quad (4.17.8)$$

for all measurable functions f .

In view of (4.17.8), the de la Vallée–Poussin means

$$V_n(w, f) = n^{-1} \sum_{k=n+1}^{2n} S_k(w, f) \quad (4.17.9)$$

provide approximation with rate equivalent to the best one by n th-degree polynomials.

We already know from Section 4.3 that Lebesgue functions can be estimated in terms of Christoffel functions, and the former are used to prove convergence of orthogonal Fourier series under various smoothness conditions. On the other hand, trigonometric Fourier series of functions of bounded variation do converge even though the functions might not be smooth at all (cf. [Zy1, p. 57]). Naturally, if one has a proper equiconvergence theorem such as Theorem 4.12.5, then convergence of orthogonal Fourier series follows from that of trigonometric Fourier series. Otherwise, one needs to treat and judge each orthogonal system on its own merit.

The soft proof of convergence of trigonometric Fourier series of functions of bounded variation is using Littlewood's Tauberian theorem [Li] (cf. [Zy1, p. 81]), which guarantees convergence of Abel summable series with $O(1/n)$ terms. What Freud [Fr57, Theorem 3.7, p. 118] noticed was that this approach was perfectly fit for orthogonal Fourier series associated with Freud-type weights.

THEOREM 4.17.2 [Fr57]. *Let w be defined by (4.17.1), where Q satisfies conditions (4.17.5)–(4.17.7). Let f be continuous on \mathbb{R} and of bounded variation in every finite interval and let*

$$\int_{\mathbb{R}} w(t)^{1/2} |df(t)| < \infty. \quad (4.17.10)$$

Then

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} w(x)^{1/2} |f(x) - S_n(w, f, x)| = 0. \quad (4.17.11)$$

Sketch of Proof of Theorem 4.17.2. First we express $S_n(w, f, x)$ in terms of the de la Vallée–Poussin means $V_n(w, f, x)$ as

$$S_n(w, f, x) = V_n(w, f, x) - \sum_{k=n+1}^{2n-1} [2 - (k/n)] c_k(w, f) p_k(w, x), \quad (4.17.12)$$

where $c_k(w, f)$ denote the Fourier coefficients of f (cf. (3.11)). Hence

$$|S_n(w, f, x) - V_n(w, f, x)| \leq \sum_{k=n+1}^{2n-1} |c_k(w, f) p_k(w, x)|, \quad (4.17.13)$$

and by Schwarz' inequality

$$|S_n(w, f, x) - V_n(w, f, x)|^2 \leq \sum_{k=n-1}^{\infty} |c_k(w, f)|^2 \sum_{k=0}^{2n-1} p_k(w, x)^2, \quad (4.17.14)$$

i.e.,

$$|S_n(w, f, x) - V_n(w, f, x)|^2 \leq E_n(w, f, 2)^2 \lambda_{2n}(d\alpha, x)^{-1}, \quad (4.17.15)$$

where $E_n(w, f, 2)$ denotes the best $L_2(w)$ approximation of f on the real line. Now we can apply Theorem 4.16.6 to estimate the reciprocal of the Christoffel function on the right-hand side of (4.17.15), and we obtain

$$w(x) |S_n(w, f, x) - V_n(w, f, x)|^2 \leq \text{const}(n/q_n) E_n(w, f, 2)^2, \quad (4.17.16)$$

where q_n is the solution of $qQ'(q) = n$. By Theorem 4.17.1,

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} w(x)^{1/2} |f(x) - V_n(w, f, x)| = 0. \quad (4.17.17)$$

The next step in the proof is to show

$$\lim_{n \rightarrow \infty} (n/q_n) E_n(w, f, 2)^2 = 0. \quad (4.17.18)$$

Here I will skip the details. I just note that (4.7.18) is proved by first estimating $E_n(w, f, 2)$ in terms of $E_n(w^{1/2}, f, 1)$ and $E_n(w^{1/2}, f, \infty)$, where

$E_n(w^{1/2}, f, p)$ denotes the best $L_p(w^{1/2})$ approximation of f , and then estimating $E_n(w^{1/2}, f, 1)$ and $E_n(w^{1/2}, f, \infty)$, using Theorem 4.17.1 and some other approximation techniques developed by Freud in a number of papers such as [Fr50; Fr54; Fr58; Fr57, Appendix, p. 119] (cf. [Mh4, DiTo1, DiTo2, DiLuNeTo]). The latter techniques involve Bohr-type inequalities and one-sided approximation of Heaviside's Γ_x function (4.2.3) by polynomials on the whole real line in a way that resembles inequalities (4.2.4)–(4.2.5). Now (4.17.11) directly follows from (4.17.16)–(4.17.18). ■

H. N. Mhaskar [Mh3] proved a number of related results regarding orthogonal Fourier series of functions of bounded variation.

The term “Bernstein–Markov inequalities” refers to estimate of norms of derivatives of polynomials in one Banach space in terms of norms of polynomials in possibly another Banach space, and their generalizations to metric spaces. The classical Markov inequality states

$$\|II'_n\|_c \leq n^2 \|II_n\|_c, \quad (4.17.19)$$

whereas, according to Bernstein's inequality,

$$\|uII'_n\|_c \leq n \|II_n\|_c, \quad u(x) = (1 - x^2)^{1/2}, \quad (4.17.20)$$

for all algebraic polynomials II_n of degree at most n , where $\|\cdot\|_c$ denotes the maximum norm in $[-1, 1]$ (cf. [Be5, pp.13–27; Nat, Vol. I, pp. 90, 133, 137]). Bernstein–Markov inequalities are of invaluable help in characterizing smoothness of functions in terms of the rate of their best approximations in one or another space. As a matter of fact, such problems are usually resolved by arguments that are either identical to or close imitations of Bernstein's proofs in [Be5, pp. 28–41].

Freud was very well aware of the need for Bernstein–Markov inequalities in $L_p(w)$ spaces with Freud weights (and so was Szegő (cf. [Sz4, pp. 845–851]), and such inequalities appeared at an early stage of his attempts to establish a theory of best approximation on infinite intervals. His first Bernstein–Markov inequality was in $L_\infty(w)$, where w is the Hermite weight [Fr36, Theorem 1, p. 109], and he soon generalized his results to all $L_p(w)$ spaces in [Fr40, Theorem 1, p. 570] as follows.

THEOREM 4.17.3 [Fr40]. *Let w be defined by*

$$w(x) = \exp(-x^2/2), \quad x \in \mathbb{R}, \quad (4.17.21)$$

and let $1 \leq p \leq \infty$. Then there exists a constants $c = c(p)$ such that

$$\|II'_n w\|_p \leq cn^{1/2} \|II_n w\|_p \quad (4.17.22)$$

for all polynomials $II_n \in \mathbb{P}_n$, where $\|\cdot\|_p$ denotes the L_p norm in \mathbb{R} .

There is no doubt that (4.17.22) is both beautiful and significant. Nevertheless, Freud committed two unfortunate sins with this theorem.

Sin No. 1. Freud was unaware of W. E. Milne's paper [Mi2], where

$$\|(II_n w)'\|_\infty \leq cn^{1/2} \|II_n w\|_\infty \quad (4.17.23)$$

is proved, which is essentially the same as (4.17.22) with $p = \infty$. The paper [Mi2] was published in the *Transactions of the American Mathematical Society*, which is easily available. Moreover, another paper of W. E. Milne—on approximation theorems over infinite intervals [Mi1]—was quoted in D. Jackson's monograph [Ja, p. 108], in which an entire section is devoted to such problems (cf. [Ja, Sect. 3.5, pp. 101–108]). I do not deny my ignorance either, and I thank R. A. Zalik for bringing [Mi2] to my attention (cf. [Za]). I discovered [Ja, p. 108] only several years after Freud started to produce his Bernstein–Markov inequalities.

Sin No. 2. This refers to the method of proof which Freud later kept as a model for all of his Bernstein–Markov inequalities (cf. [Fr44, Fr48, Fr53, Fr66, Fr69]) and which turned out to be not just overly complicated but also obstructing the way to proper generalizations. I will briefly elaborate on the

Sketch of Proof of Theorem 4.17.3

This proof consists of four parts.

Step 1. Freud first proves

$$\|1_n II_n' w\|_\infty \leq cn^{1/2} \|1_n II_n w\|_\infty \quad (4.17.24)$$

(cf. [Fr36, p. 112]) where 1_n denotes the characteristic function of the interval $[-\frac{3}{2}n^{1/2}, \frac{3}{2}n^{1/2}]$. This is verified by repeating Bernstein's arguments (cf. [Nat, Vol. I, pp. 90–92]) for a beautiful and clear exposition adapted to the case of Hermite weights, with Hermite polynomials taking over the role of the trigonometric functions $\sin(nt)$.

Step 2. Now (4.17.22), with $p = \infty$, follows from (4.17.24) and Freud's infinite–finite range inequality

$$\|II_n w\|_\infty \leq c \|1_n II_n w\|_\infty \quad (4.17.25)$$

(cf. [Fr36, p. 109] and Theorem 4.16.3).

Step 3. Using (4.17.25), Freud shows

$$\|II_n' w\|_1 \leq cn^{1/2} \|II_n w\|_1 \quad (4.17.26)$$

by applying duality arguments. Here the reasoning goes as follows (cf. [Fr40, Lemma 3, p. 571]). We have

$$\begin{aligned} \|I'_n w\|_1 &= \sup_{\|g w\|_\infty \leq 1} \int_{\mathbb{R}} g(t) I'_n(t) w(t)^2 dt \\ &= \sup_{\|g w\|_\infty \leq 1} \int_{\mathbb{R}} V_n(w^2, g, t) I'_n(t) w(t)^2 dt, \end{aligned} \quad (4.17.27)$$

where $V_n(w^2, g)$ denotes the de la Vallée–Poussin sum (4.17.9) associated with the Hermite weight $w(t)^2 = \exp(-t^2)$. Integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} V_n(w^2, g, t) I'_n(t) w(t)^2 dt \\ &= - \int_{\mathbb{R}} V'_n(w^2, g, t) I_n(t) w(t)^2 dt + 2 \int_{\mathbb{R}} V_n(w^2, g, t) I_n(t) t w(t)^2 dt. \end{aligned} \quad (4.17.28)$$

Therefore, by (4.17.24), Theorem 4.17.1 (with $p = \infty$) and the infinite–finite range inequality (4.16.11) (with $p = 1$),

$$\left| \int_{\mathbb{R}} V_n(w^2, g, t) I_n(t)' w(t)^2 dt \right| \leq cn^{1/2} \|g w\|_\infty \|I_n w\|_1, \quad (4.17.29)$$

and thus (4.17.26) follows from (4.17.27) and (4.17.29).

Step 4. Since we have already proved (4.17.22) for $p = 1$ and $p = \infty$, by Theorem 4.17.1,

$$\|V'_n(w^2, g) w\|_p \leq cn^{1/2} \|V_n(w^2, g) w\|_p \leq cn^{1/2} \|g w\|_p \quad (4.17.30)$$

for $p = 1$ and $p = \infty$. By Riesz and Thorin's interpolation theorem (cf. [Zy2, p. 93]), (4.17.30) remains valid for all $1 < p < \infty$. Noting that $V_n(w^2, g)$ acts as a projector on \mathbb{P}_n (cf. (4.17.9)), Freud's Bernstein–Markov inequality (4.17.22) follows from (4.17.30). ■

What Freud's proof of Theorem 4.17.3 missed was that it could and should have been proved via infinite–finite range inequalities with the well-known L_p version of Bernstein's inequality (4.17.20) (cf. [Zy2, p. 11]) taken as starting point. This I noticed in the mid-seventies and it was first published in S. S. Bonan's Ph.D. dissertation [Bon1]. My simplification of Freud's proof of (4.17.22) and related inequalities was subsequently resurrected by A. L. Levin and D. S. Lubinsky in [LevLu1, LevLu2]. I will introduce the reader to these ideas by providing an outline for a proof which, as a matter of fact, works for all $0 < p \leq \infty$.

Sketch of the Right Proof of Theorem 4.17.3. The essence of the proof is that, on every interval $[-c_1 n^{1/2}, c_1 n^{1/2}]$, the weight function w in (4.17.21) can be approximated by polynomials R_n of degree $c_2 n$ so that

$$w(x) \sim R_n(x), \quad |x| \leq c_1 n^{1/2}, \quad (4.17.31)$$

and

$$|R'_n(x)| \leq c_3 n^{1/2} w(x), \quad |x| \leq c_1 n^{1/2}. \quad (4.17.32)$$

The construction of such polynomials goes back to Freud [Fr51], who used partial sums of the Taylor series of w to find R_n with the required properties. By the infinite-finite range inequality (4.16.19), there exists a constant c_1 such that

$$\|I'_n w\|_p \leq c \|1_n I'_n w\|_p, \quad (4.17.33)$$

where 1_n denotes the characteristic function of the interval $[-c_1 2^{-1} n^{1/2}, c_1 2^{-1} n^{1/2}]$. Here and in what follows $\|\cdot\|_p$ means the p th root of the integral of the p th power of the absolute value, which is of course not a norm for $0 < p < 1$. By (4.17.31) we obtain

$$\begin{aligned} \|I'_n w\|_p &\leq c \|1_n I'_n R_n\|_p = c \|1_n [(I_n R_n)' - I_n R'_n]\|_p \\ &\leq c \|1_n (I_n R_n)'\|_p + c \|1_n I_n R'_n\|_p. \end{aligned} \quad (4.17.34)$$

By the L_p version of Bernstein's inequality, for every $0 < p < \infty$, there is a constant c such that

$$\|1_{[-1,1]} r'_n\|_p \leq cn \|1_{[-2,2]} r_n\|_p \quad (4.17.35)$$

for every polynomial $r_n \in \mathbb{P}_m$, $m \leq \text{const} \cdot n$, where $1_{[a,b]}$ denotes the characteristic function of $[a, b]$ (cf. [Ar, MáNe1, Ne21]). A change of variables transforms (4.17.35) to

$$\|1_n r'_n\|_p \leq cn^{1/2} \|1_n r_n\|_p, \quad (4.17.36)$$

where 1_n denotes the characteristic function of $[-c_1 n^{1/2}, c_1 n^{1/2}]$. Now we can apply (4.17.36) to the first term on the right-hand side of (4.17.34) and we obtain

$$\|I'_n w\|_p \leq cn^{1/2} \|1_n I_n R_n\|_p + c \|1_n I_n R'_n\|_p. \quad (4.17.37)$$

Finally, by (4.17.31) and (4.17.32), we can estimate R_n and R'_n in terms of w , and thus (4.17.22) follows from (4.17.37). ■

For wide classes of Freud weights the latter approach makes Bernstein-Markov inequalities much easier to prove, and it also enables

one to prove such inequalities when Freud's original method stops functioning. For instance, Freud's method cannot handle $L_p(w)$ spaces with $0 < p < 1$.

One important class of weights for which Freud did not prove Bernstein–Markov inequalities is

$$w(x) = \exp(-|x|^m), \quad x \in \mathbb{R}, \quad (4.17.38)$$

for $0 < m < 2$. In recent work of A. L. Levin and D. S. Lubinsky [LevLu1, LevLu2] ($1 < m < 2$) and V. Totik and mine [NeTo1] ($0 < m \leq 1$), this problem has been completely resolved as follows.

THEOREM 4.17.4 [LevLu1, NeTo1]. *Let w be given by (4.17.38) and let $0 < p < \infty$. Then there exists a constant $c = c(p, m)$ such that, for every polynomial $\Pi_n \in \mathbb{P}_n$,*

$$\|\Pi'_n w\|_p \leq cn^{1-1/m} \|\Pi_n w\|_p \quad (4.17.39)$$

if $m > 1$,

$$\|\Pi'_n w\|_p \leq c \log n \|\Pi_n w\|_p \quad (4.17.40)$$

if $m = 1$, and

$$\|\Pi'_n w\|_p \leq c \|\Pi_n w\|_p \quad (4.17.41)$$

if $0 < m < 1$.

Nikolskii inequalities are natural extensions of Christoffel function estimates and Bernstein–Markov inequalities in the sense that they seek a relationship between metrics in different finite-dimensional metric spaces of polynomials. The first such inequality was found by S. M. Nikolskii [Ni] (cf. [Ti, p. 229]), and it deals with estimating L_q norms of trigonometric polynomials in terms of their L_p norms for $p < q$ (for $p > q$ this is trivially done by Hölder's inequality). In [Ne19, Chap. 6.3] I not only gave a variety of such results in weighted $L_p(w)$ spaces on finite intervals but also suggested a general method of attacking such problems which should be applicable in a number of settings including the case of $L_p(w)$ spaces on infinite intervals with Freud weights. In spite of my recommendation to follow my method, H. N. Mhaskar chose in [Mh1] another approach to proving Nikolskii inequalities for the above spaces. Then in their joint paper (MhSa2], Mhaskar and E. B. Saff used my method to extend and improve results of [Mh1] with simpler proofs (cf. [LevLu1]). In the particular cases of $L_p(w)$ with Hermite and Laguerre weights such inequalities were proved by C. Markett [Mark2] and R. A. Zalik [Za].

While many of the Nikolskii inequalities proved by Markett, Mhaskar, Saff, and Zalik are essentially accurate (cf. [Mh4, Theorem 8]), V. Totik and I in a recent paper [NeTo2] found the sharpest possible Nikolskii inequalities with Freud weights (4.17.38) for all $m > 0$. What pleases me the most is that our results are based on Christoffel function estimates combined with infinite–finite range inequalities (cf. Theorem 4.16.2) and methods developed in [Ne19, Chap. 6.3]. Therefore it is appropriate that they be mentioned here, even though Freud himself never dealt with such inequalities. In what follows $\|\cdot\|_p$ again means the p th root of the integral of the p th power of the absolute value.

THEOREM 4.17.5 [NeTo2]. *Let $m > 0$, and let w be given by (4.17.38). For given $0 \leq p, q < \infty$ and $n = 1, 2, \dots$, define $K_n = K_n(m, p, q)$ by*

$$\begin{aligned} K_n &= (n^{1/m})^{(1/p - 1/q)} && \text{if } p \leq q \\ K_n &= (n^{1 - 1/m})^{(1/q - 1/p)} && \text{if } p > q \quad \text{and} \quad m > 1 \\ K_n &= (\log n)^{(1/q - 1/p)} && \text{if } p > q \quad \text{and} \quad m = 1 \\ K_n &= 1 && \text{if } p > q \quad \text{and} \quad m < 1. \end{aligned} \quad (4.17.42)$$

Then there exists a constant $c = c(m, p, q) > 0$ such that

$$\|\Pi_n w\|_p \leq c K_n \|\Pi_n w\|_q \quad (4.17.43)$$

for every polynomial $\Pi_n \in \mathbb{P}_n$. Inequality (4.17.43) is best possible in the sense that, given $m > 0$, $p > 0$ and $q > 0$, there is a constant $c^* > 0$ and a sequence of polynomials $\{R_n\}$, $n = 1, 2, \dots$, such that

$$\|R_n w\|_p \geq c^* K_n \|R_n w\|_q \quad (4.17.44)$$

for $n = 1, 2, \dots$

Hint for Proof of Theorem 4.17.5. Infinite–finite range inequalities such as Theorem 4.16.2 enable one to reduce (4.17.43) to integrals over finite intervals of lengths approximately $n^{1/m}$. On such intervals, the weight function w can be approximated by polynomials; this has been accomplished in various papers by authors such as Freud, A. L. Levin, D. S. Lubinsky, V. Totik, and I (cf. [Fr54, LevLu1, LevLu2, NeTo1]). Hence (4.17.43) is further reduced to a Nikolskii inequality in a finite interval with no weight function, and such inequalities were proved in [Ne19, p. 114]. ■

An interesting point concerning the Nikolskii inequality (4.17.43) is that the order of magnitude of the constant K_n is different for $p \leq q$ and $p > q$ except for the Hermite case $m = 2$. C. Markett proved in [Mark2,

Theorem 1, p. 811] that the Laguerre case is similar to the case with the Hermite weight.

Those familiar with Freud's research on weighted approximation on infinite intervals must have observed that I have failed to discuss another favorite inequality of Freud. I refer to Bohr-type inequalities, which play a major role in Freud's Jackson-type theorems such as those given in [Fr50, Fr51, Fr54, FrNe2]. Since (i) I cannot delay elaborating on Freud's conjectures in the next section, and (ii) no claims have been made as to the completeness of this survey in any respect, even regarding topics where Christoffel functions are of crucial significance (and this is certainly the case for Bohr-type inequalities), I conclude this section by stating Harald Bohr's inequality (cf. [Bo, FrSz1, SzöSt]) and let the reader turn to original sources for Freud's results and methods in this subject.

THEOREM 4.17.6 [Bo]. *Let T be defined by*

$$T(t) = \sum_{k=-N}^N c_k \exp(i_{\rho_k} t), \quad (4.17.45)$$

where ρ_k are integers such that $\rho_k \geq n > 0$. Let

$$T^*(t) = \sum_{k=-N}^N c_k (i_{\rho_k})^{-1} \exp(i_{\rho_k} t), \quad (4.17.46)$$

antiderivative of T . Then

$$\max_{t \in \mathbb{R}} |T^*(t)| \leq (\pi/2) n^{-1} \max_{t \in \mathbb{R}} |T(t)| \quad (4.17.47)$$

and the constant $\pi/2$ is sharp.

4.18. Freud Conjectures

In his papers [Fr65, Fr68, Fr71], Freud formulated two conjectures which subsequently turned out not just to be the tour de force of his contributions to orthogonal polynomials but also to have the greatest impact of all of his work in approximation theory.

Conjecture 4.18.1 [Fr65, Fr68]. Let w be defined by

$$w(x) = \exp(-|x|^m), \quad x \in \mathbb{R}, \quad (4.18.1)$$

with $m > 1$, and let a_n denote the recursion coefficients in (3.7). Then

$$\lim_{n \rightarrow \infty} n^{-1/m} a_n = [\Gamma(2^{-1}m) \Gamma(2^{-1}m + 1) \Gamma(m + 1)^{-1}]^{1/m}. \quad (4.18.2)$$

Conjecture 4.18.2 [Fr68, Fr71]. Let w be given by (4.18.1) with $m > 1$, and let x_{1n} denote the greatest zero of the orthogonal polynomial $p_n(w)$. Then

$$\lim_{n \rightarrow \infty} n^{-1/m} x_{1n} = 2[\Gamma(2^{-1}m) \Gamma(2^{-1}m + 1) \Gamma(m + 1)^{-1}]^{1/m}. \quad (4.18.3)$$

These conjectures and the papers in which they were published have quite a history, which I will briefly describe here. As a matter of fact, Freud's interest in recursion coefficients and greatest zeros arose not because he had ever been seriously interested in three-term recurrences or quadratic forms whose norms are related to x_{1n} . What he wanted was the possibility of creating sequences of polynomials which are capable of approximating functions in weighted L_p spaces on the whole real line with rate as close to the optimal as possible. At an early stage of the game Freud decided to put his money on de la Vallée-Poussin (delayed arithmetic) sums as the means of approximations, and thus he had to manipulate orthogonal Fourier sums. Alas, according to the Christoffel–Darboux formula (3.13), partial sums of orthogonal Fourier series contain the recursion coefficient a_n as an essential ingredient (cf. (3.8)). Of course, if $\text{supp}(d\alpha)$ is compact, then the sequence $\{a_n\}$ is bounded and hence does not interfere with estimating $(C, 1)$ sums of orthogonal Fourier series. However, this is not the case if the support of the measure is no longer compact. Thus it is essential to be able to estimate the size of a_n . As an initial approach one writes

$$a_{n-1} = \int_{\mathbb{R}} x p_{n-2}(d\alpha, x) p_{n-1}(d\alpha, x) d\alpha(x), \quad (4.18.4)$$

and then, by the Gauss–Jacobi quadrature formula (3.4),

$$a_{n-1} = \sum_{k=1}^n x_{kn} p_{n-2}(d\alpha, x_{kn}) p_{n-1}(d\alpha, x_{kn}) \lambda_{kn}(d\alpha), \quad (4.18.5)$$

where $x_{kn} = x_{kn}(d\alpha)$. Hence, if $d\alpha$ is symmetric with respect to 0, then

$$a_{n-1} \leq x_{1n} \quad (4.18.6)$$

(cf. [Fr31b, Problem 1.10, p. 49], where it was printed with an error). This inequality explains why Freud became interested in greatest zeros of orthogonal polynomials. I add that, for symmetric measures $d\alpha$, $\text{supp}(d\alpha)$ is compact if and only if the recursion coefficients form a bounded sequence, and the latter holds if and only if all the zeros of the corresponding orthogonal polynomials are uniformly bounded (cf. [Ne19, Lemma 3.3.1, p. 20]).

In view of the significance that I attach to Freud's conjectures, the reader may be interested in the unusual circumstances surrounding their publication. Chronologically, the first paper is [Fr68], which Freud wrote in August 1973 and submitted on December 1, 1973, to the "Proceedings of a Colloquium on the Constructive Theory of Functions" held at Babes-Bolyai University in Cluj, Rumania, in September 1973. However, the organizer of that conference, T. Popoviciu, passed away before the publication of this book. To Freud's great surprise, the paper suddenly appeared in *Matematica, Revue d'Analyse Numérique a de Théorie de l'Approximation* without previous authorization by him. He was shocked and infuriated indeed on learning that his paper was published in 1977 in a journal to which he had no intension of submitting it. (References [7] and [8], i.e., [Fr59] and [Fr56], are given there as "in print," whereas they were actually published in 1974; and [9], i.e., [Fr65], is listed as "*Studia Sci. Math. Hungar.* (in print).") Let me add that I had an identical experience with a paper which I submitted to the same conference proceedings.

In one respect, these references were right: [Fr65] was indeed submitted to *Studia Scientiarum Mathematicarum Hungaricae* in January 1974. However, when, in August 1974, Freud left Hungary and became a free agent for a while, he withdrew his paper from that journal (it might have been rejected, of course; we will perhaps never find this out), and subsequently he submitted it to the *Proceedings of the Royal Irish Academy* on November 4, 1974. Why did he choose this journal? I have frequently been asked this question by friends and colleagues. Here is the answer: being a homeless refugee, Freud was desperately seeking a permanent position and residence, which were necessary conditions for him to be able to get his family out of Hungary. For a while it looked as though he was going to stay in Ireland. In order to introduce himself and demonstrate his good will, Freud read (Fr65) before the Royal Irish Academy since this was his only paper available at the time. This is the story of [Fr65], to which I add that Freud remained grateful to the Irish for the rest of his life, although he eventually chose the United States as home.

The story behind the late appearance of [Fr71] is less romantic. In March 1978, Freud organized a special session on orthogonal polynomials at a meeting of the American Mathematical Society at the Ohio State University in Columbus, Ohio. For this session, Freud prepared a talk titled "On the Greatest Zero of an Orthogonal Polynomial" which was subsequently followed up by a paper with the same title. The original version of this paper, however, was not accepted for publication. This was due to lack of organization in the exposition. Freud's untimely death prevented him from revising it, and thus the task of improving the presentation fell upon me. On the occasion of the *Journal of Approximation Theory's* Freud

memorial volumes I finally completed a publishable version of Freud's paper and the final product appears as [Fr71].

If w in (4.18.1) is the Hermite weight ($m=2$), then the corresponding recurrence coefficients are given by

$$a_n = (n/2)^{1/2}, \quad (4.18.7)$$

and thus Freud's Conjecture (4.18.2) is obvious. For $m=4$, the recurrence coefficients a_n are the unique solutions of

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2), \quad a_0^2 = 0, n = 1, 2, \dots \quad (4.18.8)$$

(cf. [Fr65] for the equation and [LewQu; Ne29, Theorem 3, p. 268] for the uniqueness of the solution). As Freud noticed in [Fr65], one can prove (4.18.2) by playing around with limit inferior and limit superior of $a_n n^{-1/4}$ in (4.18.8). In the same paper, [Fr65], Freud proved (4.18.2) if $m=6$ in (4.18.1), via application of this simple ad hoc method. In the latter case, the recurrence coefficients satisfy

$$n = 6a_n^2(a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 + 2a_{n+1}^2 a_n^2 + a_{n+1}^2 a_{n-1}^2 + a_n^4 + 2a_n^2 a_{n-1}^2 + a_{n-1}^4 + a_{n-1}^2 a_{n-2}^2), \quad a_{-1} = 0, a_0 = 0, n = 1, 2, \dots \quad (4.18.9)$$

(cf. [Fr65, Ma4, MáNe5, Sh1, Sh2]). Here again (4.18.9) has a unique positive solution (cf. [Ma6]).

If m is even, then the weight in (4.18.1) is such that w'/w is a polynomial. This observation enables one to obtain a recursive formula for the corresponding recurrence coefficients as follows. By orthogonality,

$$n/a_n = \int_{\mathbb{R}} [p_{n-1}(w, x) p_n(w, x)]' w(x) dx \quad (4.18.10)$$

so that integration by parts leads to

$$n/a_n = m \int_{\mathbb{R}} x^{m-1} p_{n-1}(w, x) p_n(w, x) w(x) dx. \quad (4.18.11)$$

Repeatedly applying the recurrence formula (3.7), one obtains the orthogonal Fourier expansion of $x^{m-1} p_{n-1}(w, x)$ in terms of $p_k(w, x)$ and a_k and then the integral on the right-hand side of (4.18.11) is nothing but the n th Fourier coefficient. A somewhat more sophisticated point of view identifies the right-hand side of (4.18.11) as the $(n-1, n)$ th entry in the matrix $[\log w(A)]'$, where A is the Jacobi matrix given by

$$A = \left[\int_{\mathbb{R}} x p_j(w, x) p_k(w, x) w(x) dx \right] \quad (4.18.12)$$

(cf. [Fr65; Ma4; Ma6; MáNeZa; Ne19, Lemma 4.2.12, p. 45]). These observations enable one to find the recurrence coefficients a_n from

$$n/a_n = P(a_{n+k}; k = -2^{-1}m + 1, -2^{-1}m + 2, \dots, 2^{-1}m - 1) \quad (4.18.13)$$

(m even and w given by (4.18.1)) with suitable initial conditions, where P is a homogeneous polynomial in the variables of degree m (cf. [MáNeZa]).

What differentiates the cases $m=2, 4$ and 6 from $m=8, 10, \dots$ is that formulas (4.18.7)–(4.18.9) are center weighted, whereas (4.18.13) is not. By center weightedness I mean that the coefficients a_n figure with more frequency in (4.18.7)–(4.18.9) than a_{n+k} with $k \neq 0$. In other words, the associated Jacobian is such that the diagonal elements dominate the matrix, and hence invertibility becomes a simple matter of fact. A closer examination of Freud's [Fr65] shows that this is exactly the reason why his method yields (4.18.2) for $m=4$ and 6 . In my paper with A. Máté and T. Zaslavsky [MáNeZa] we show by a combinatorial argument that this is no longer true for $m \geq 8$, and thus Freud's Conjecture 4.18.1 cannot be true for obvious reasons (in other words, if it is true, then it is so for reasons deeper than obvious).

The first breakthrough towards, settling Conjecture 4.18.1 was made by H. N. Mhaskar and E. B. Saff [MhSa2] and E. A. Rahmanov [Rah5, Rah6], where (4.18.2) was proved in the sense of geometric means. I find it interesting to point out that Rahmanov not only worked independently of Mhaskar and Saff but was also apparently unaware of Freud's Conjectures and the relevant research. While Rahmanov could only treat $m > 1$ in (4.18.1), Mhaskar and Saff's methods yield characterization of the recurrence coefficients for all $m > 0$. In all fairness, I have to point that Rahmanov's weights are more general than those in (4.18.1).

Then came Alphonse Magnus, who, in December 1983, proved

THEOREM 4.18.3 [Ma4]. *Let m be an even positive integer and let w be defined by (4.18.1). Then (4.18.2) holds.*

Magnus' proof is based on the positive definiteness of the Jacobian associated with (4.18.13) which comes from $[\log w(A)]'$, where the Jacobi matrix A is given by (4.18.12). In my paper [Ne36] I suggested some slight improvements of his method which subsequently led to the following strengthening of Theorem 4.18.3 by Magnus in [Ma6], which was produced specifically for the Freud memorial volumes of the *Journal of Approximation Theory*.

THEOREM 4.18.4 [Ma6]. *Let m be a positive even integer and let w be given by*

$$w(x) = \exp(-x^m + \Pi_{m-1}(x)), \quad x \in \mathbb{R}, \quad (4.18.14)$$

where Π_{m-1} is a polynomial of degree at most $m-1$. Let a_n and b_n denote the corresponding recurrence coefficients in (3.7). Then (4.18.2) and

$$\lim_{n \rightarrow \infty} n^{-1/m} b_n = 0 \quad (4.18.15)$$

hold.

I have no doubt whatsoever that Theorem 4.18.4 is done of the most magnificent developments in the recent history of orthogonal polynomials associated with exponential weights. I am well aware of the great communal effort that went into proving it, and it was Magnus' great accomplishment that he succeeded where so many of us failed.

Since, for $m=2$, we have not only (4.18.2) but also the more accurate (4.18.7), one may well speculate about the rate of convergence in (4.18.2) for other values of m . It turns out that it is possible to obtain such estimates. This was subject to investigations by J. S. Lew and D. A. Quarles [LewQu] ($m=4$), A. Máté and me [MáNe5] ($m=6$), and A. Máté, T. Zaslavsky, and me [MáNeZa] (m even).

THEOREM 4.18.5 [MáNeZa]. *Let m be an even positive integer and let w be given by (4.18.1). Then $n^{-1/m} a_n$ has an asymptotic expansion*

$$n^{-1/m} a_n \sim \sum_{j=0}^{\infty} c_j n^{-2j}, \quad (4.18.16)$$

where

$$c_0 = [\Gamma(2^{-1}m) \Gamma(2^{-1}m+1) \Gamma(m+1)^{-1}]^{1/m}. \quad (4.18.17)$$

The proof of Theorem 4.18.5 is based on Magnus' Theorem 4.18.3. In [MáNe5] we show that convergent solutions of a smooth recurrence equation whose gradient satisfies a certain "nonunimodularity" condition can be approximated by an asymptotic expansion. The lemma used to show this has some features in common with Poincaré's theorem [Po] on homogeneous linear difference equations. In [MáNeZa] we solve a combinatorial enumeration problem concerning a one-dimensional lattice walk, and this is applied to show that the recurrence coefficients in (3.7) associated with w in (4.18.1) with $m=2, 4, \dots$ are solutions of the smooth difference equation (4.18.13) satisfying the above-mentioned "nonunimodularity" condition, and thereby (4.18.16) is verified. The "nonunimodularity" condition can be summarized as

THEOREM 4.18.6 [MáNeZa]. *Let m be an even positive integer and let w*

be given by (4.18.1). Let the polynomial P be defined by the recursion formula (4.18.13). Let z be an arbitrary complex number with $|z| \neq 1$. Then

$$\sum_{l=-m/2}^{m/2} z^l \partial(x_0 P(x_k; k = -2^{-1}m + 1, -2^{-1}m + 2, \dots, 2^{-1}m - 1)) / \partial x_l \neq 0 \tag{4.18.18}$$

provided $x_k = 1$ for all k .

W. C. Bauldry's recent results in his Ph.D. thesis [Bau2, Theorem 2.3.3, p. 36] combined with our joint research with A. Máté show that one is very close to being able to generalize Theorem 4.18.5 to weights of the form (4.18.14) (cf. [BauMáNe]).

Zeros of orthogonal polynomials are eigenvalues of truncated Jacobi matrices (cf. (4.18.12)), and thus it is clear that if the measure $d\alpha$ is symmetric around the origin (that is, if all recurrence coefficients b_n in (3.7) vanish), then the greatest zero $x_{1n}(d\alpha)$ of $p_n(d\alpha)$ can be expressed in terms of $a_n(d\alpha)$ in (3.7) by

$$x_{1n}(d\alpha) = 2 \max_{j_k \geq 0} \left[\sum_{k=1}^n a_{n-k}(d\alpha) j_k j_k + 1 \middle/ \sum_{k=1}^n j_k^2 \right] \tag{4.18.19}$$

(cf. [Fr71; MáNeTo3; Sz2, p. 186]). Thus any information regarding asymptotic behavior of the recurrence coefficients can be turned into estimates of greatest zeros of orthogonal polynomials, although this is not necessarily a reversible process. In particular, (4.18.3) is an easy consequence of (4.18.2).

Contrary to all expectations, however, it was Conjecture 4.18.2 of Freud which was first settled in its entire generality. In his brilliant paper [Rah6, Lemma 11, p. 182], Rahmanov proved

THEOREM 4.18.7 [Rah6]. *Let w be a (not necessarily even) weight function on the real line, and assume that there exists $m > 1$ such that*

$$\lim_{x \rightarrow \infty} |x|^{-m} \log w(x) = -1. \tag{4.18.20}$$

Let $X_n = \max |x_{kn}(w)|, k = 1, 2, \dots, n$. Then

$$\lim_{n \rightarrow \infty} n^{-1/m} X_n = 2 [\Gamma(2^{-1}m) \Gamma(2^{-1}m + 1) \Gamma(m + 1)^{-1}]^{1/m}. \tag{4.18.21}$$

Although Freud's Conjecture 4.18.1 has not been established yet for all $m > 1$, it is safe to say that at the present time more has been achieved than Freud would have dreamed in connection with this conjectures. I per-

sonally value these conjectures so much because they generated renewed interest in orthogonal polynomials among experts all over the world. In what follows I discuss results whose solution would have been impossible without Freud's conjectures in one sense or another.

It has been known for some time that distribution properties of zeros of orthogonal polynomials strongly depend on the behavior of the recurrence coefficients a_n and b_n in (3.7). This is a natural phenomenon since zeros of orthogonal polynomials can be identified with eigenvalues of finite sections of Jacobi matrices (cf. (4.8.12)). Such a relationship is explored in Theorem 4.9.2, and further results are proved in [Ne19, Ne24]. For measures supported on noncompact sets the first results in this direction were obtained in [NeDe], where we considered orthogonal polynomials whose recurrence coefficients behave in a regular fashion such as the ones given by (4.18.2) and (4.18.15).

THEOREM 4.18.8 [NeDe]. *Let $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nondecreasing function such that, for every fixed $t \in \mathbb{R}$,*

$$\lim_{x \rightarrow \infty} \varphi(x+t)/\varphi(x) = 1. \quad (4.18.22)$$

Let $d\alpha$ be a given measure on the real line, and assume that there exist two numbers a and b such that the associated recurrence coefficients a_n and b_n in (3.7) satisfy

$$\lim_{n \rightarrow \infty} 2a_n/\varphi(n) = a \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n/\varphi(n) = b. \quad (4.18.23)$$

Let x_{kn} denote the zeros of the corresponding orthogonal polynomials. Then for every nonnegative integer M ,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n [x_{kn}]^M \left[\int_0^n \varphi(t)^M dt \right]^{-1} = K_M(a, b), \quad (4.18.24)$$

where K_M is defined by

$$K_M(a, b) = b^M \quad (4.18.25)$$

for $a=0$, and

$$K_M(a, b) = \pi^{-1} \int_{b-a}^{b+a} t^M [a^2 - (t-b)^2]^{-1/2} dt \quad (4.18.26)$$

for $a > 0$.

If $\varphi(t) = t^{1/m}$, then (4.18.24) takes the form

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n [x_{kn} n^{-1/m}]^M = (Mm^{-1} + 1)^{-1} K_M(a, b), \quad (4.18.27)$$

and what remains to be done is the evaluation of the measure whose moments are given by $(Mm^{-1} + 1)^{-1} K_M(a, b)$. It was J. L. Ullman [U14] who first succeeded in finding this measure. His results were greatly improved by Mhaskar and Saff [MhSa2] and Rahmanov [Rah5, Rah6], who independently of each other obtained a variety of interesting results concerning contracted zero distribution of orthogonal polynomials associated with exponential (Freud type) weights. For the sake of curiosity I add that although the authors were mutually unaware of each others research, the methods applied have common roots in potential theory, an approach developed and cultivated by Ullman [U11–U18]. Naturally, if Freud's Conjecture 4.18.1 holds, then on the basis of Theorem 4.18.7, one can easily find weak limits of contracted zero measures of Freud weights. What is most pleasing is that such results were obtained without using (4.18.2). My favorite theorem is the following, proved by Rahmanov [Rah6, Theorem 4, p. 185].

THEOREM 4.18.9 [Rah6]. *Let w be a (not necessarily even) weight function on the real line, and assume that there exists $m > 1$ such that*

$$\lim_{x \rightarrow \infty} |x|^{-m} \log w(x) = -1. \quad (4.18.28)$$

Then, for every continuous f on the real line, we have

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n f(R_m x_{kn} n^{-1/m}) = \int_{-1}^1 f(t) a_m(t) dt, \quad (4.18.29)$$

where

$$R_m = 2^{-1} [r(2^{-1}m) r(2^{-1}m + 1) r(m + 1)^{-1}]^{-1/m} \quad (4.18.30)$$

and

$$a_m(t) = mt^{m-1} \pi^{-1} \int_t^1 x^{-m} (1-x^2)^{-1/2} dx. \quad (4.18.31)$$

The M th moments of the density a_m of the limit measure of the contracted zero measures are precisely $(Mm^{-1} + 1)^{-1} K_M(1, 0)$ (cf. (4.18.26)), which was introduced in [NeDe] and which Ullman evaluated in [U14] for even integer values of m . The measure generated by a_m was named

Ullman measure by Mhaskar and Saff [MhSa2]. Ullman conjectured a wonderful characterization of all such measures in [UI9].

The density function a_m was successfully evaluated in terms of hypergeometric functions by Mhaskar and Saff in [MhSa2, p. 206] and by Rahmanov in [Rah6, p. 185]. The formula for a_m is

$$a_m(t) = \pi^{-1}(1-t^2)^{1/2} m(m-1)^{-1} {}_2F_1(1, 1-2^{-1}m; 2^{-1}3-2^{-1}m; t^2) \\ + \pi^{-1/2} t^{m-1} \tan(m\pi/2) r(2^{-1}m+1) r((m+1)/2)^{-1} \quad (4.18.32)$$

if m is not an odd positive integer, and

$$a_m(t) = \pi^{-1}(1-t^2)^{1/2} \sum_{k=0}^{p-1} [(-2^{-1}-p)_{k+1}/(-p)_{k+1}] t^{2k} \\ + \pi^{-1} [m!/p!] 2^{1-p} t^{2p} \log [(1+(1-t^2)^{1/2})/|t|] \quad (4.18.33)$$

if $m = 2p + 1$ is an odd positive integer.

Having read one of the first drafts of this work, R. Askey noticed that one can find a simpler formula for $a_m(t)$ than (4.18.32)–(4.18.33) by first introducing a few changes of the variables in the integral in (4.18.31), and then expanding into series the resulting integrands and applying Euler's transformation. Askey's formula is

$$a_m(t) = \pi^{-1}(1-t^2)^{1/2} m {}_2F_1(1, 1-2^{-1}m; \frac{3}{2}; 1-t^2). \quad (4.18.33)$$

In their recent paper [GonRa2], A. A. Goncar and Rahmanov extended 4.18.9 to a general class of measures via application of potential theoretic methods.

In what follows I discuss asymptotics with remainder terms for zeros of orthogonal polynomials. In view of (4.18.19), it is natural to expect that any asymptotic expansion of the recurrence coefficients a_n such as (4.18.16) should result in appropriate asymptotic series for the greatest zeros x_{1n} . Although there are no general theorems of this nature and flavor yet, it is clear that the relationship between zeros of orthogonal polynomials and recurrence coefficients is more than skin deep. For instance, for the Hermite weight function $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$, it is well known that the greatest zero x_{1n} of the Hermite polynomials satisfies

$$n^{-1/2} x_{1n} = 2^{1/2} - 2^{-1/2} 3^{-1/3} i_1 n^{-2/3} + o(n^{-2/3}), \quad (4.18.34)$$

where

$$i_1 < i_2 < \dots \quad (4.18.35)$$

denote the real zeros of Airy's function A which is defined as the unique solution of the differential equation

$$y'' + xy/3 = 0, \quad (4.18.36)$$

which remains bounded as $x \rightarrow -\infty$ (cf. [Sz2, p. 18]).

The usual way of obtaining asymptotics such as (4.18.34) is to use Sturm-type comparison theorems applied to the differential equation satisfied by the corresponding orthogonal polynomials (cf. [Sz2, Sect. 6.31; Ol]). However, orthogonal polynomials generated by three-term recurrence equations (cf. (3.7)) do not normally satisfy any reasonably simple differential equation, and if they do (cf. Section 4.20), then the nature of the differential equation is not always suitable for Sturm-type theorems. Thus the right approach is to treat greatest zeros of orthogonal polynomials as greatest eigenvalues of truncated Jacobi matrices and/or quadratic forms (cf. (4.18.19)). This point of view enables one to draw a parallel between eigenvalues of different quadratic forms by comparing their corresponding coefficients. In [MáNeTo3], Máté, Totik, and I used a philosophy based on the above principle to prove

THEOREM 4.18.10 [MáNeTo3]. *Let $d\alpha$ be a measure on the real line which is symmetric around the origin, and assume that the recurrence coefficients a_n in (3.7) satisfy*

$$a_n = cn^\delta [1 + o(n^{-2/3})], \quad (4.18.37)$$

where $c > 0$ and $\delta > 0$ are independent of n . Let x_{1n} denote the greatest zero of $p_n(d\alpha)$. Then

$$n^{-\delta} x_{1n} = 2c - c3^{-1/3} (2\delta)^{2/3} i_1 n^{-2/3} + o(n^{-2/3}), \quad (4.18.38)$$

where i_1 is the least zero of Airy's function A in (4.18.35)–(4.18.36).

In another recent paper [MáNeTo11] we apply analogous ideas combined with H. Weyl and R. Courant's famous theorem on eigenvalues of quadratic forms (cf. [GrSz, p. 32]) to obtain asymptotics such as (4.18.38) for all zeros x_{kn} (k fixed) of orthogonal polynomials whose recurrence coefficients satisfy (4.18.37). Our extension of Rahmanov's Theorem 4.18.7 for Freud weights (4.18.1) with m even is the following

THEOREM 4.18.11 [MáNeTo11]. *Let m be a positive even integer, and let $w(x) = \exp(-x^m)$, $x \in \mathbb{R}$. Let*

$$x_{1n} > x_{2n} > \cdots \quad (4.18.39)$$

be the zeros of $p_n(w)$. Then for all fixed values of $k = 1, 2, \dots$,

$$n^{-1/m} x_{kn} = [r(2^{-1}m) r(2^{-1}m + 1) r(m + 1)^{-1}]^{1/m} \\ \times [2 - 2^{2/3} 3^{-1/3} m^{-2/3} i_k n^{-2/3}] + o(n^{-2/3}) \quad (4.18.40)$$

as $n \rightarrow \infty$, where i_k are the zeros of Airy's function A (cf. (4.18.35) and (4.18.36)).

Needless to say, I fully expect (4.18.40) to remain valid for every $m > 1$.

One of my all-time favorite results is also based on the asymptotic estimates given in Theorem 4.18.5. In [Ne35] I combined (4.18.16) with Freud's Theorem 4.16.6 on lower bounds of Christoffel functions (yes my reader, Christoffel functions are back again) and with an ingenious formula of U. M. Dombrowski and G. M. Fricke [DoFr]. The resulting product yields sharp bounds for orthogonal polynomials associated with w in (4.18.1) for m even. Dombrowski and Fricke's formula was subsequently generalized by Dombrowski in [Do4] as follows.

THEOREM 4.18.12 [Do4]. *Let $\{p_n\}$ ($n = 0, 1, \dots$) be an arbitrary system of orthogonal polynomials, and let $\{a_n\}$ and $\{b_n\}$ denote the recursion coefficients in (3.7)–(3.8). Let S_n be defined by*

$$S_n(x) = \sum_{k=0}^n \{ [a_{k+1}^2 - a_k^2] p_k(x)^2 + a_k [b_k - b_{k-1}] p_{k-1}(x) p_k(x) \}. \quad (4.18.41)$$

Then we have

$$S_n(x) = a_{n+1}^2 [p_n(x)^2 - a_{n+1}^{-1} (x - b_n) p_n(x) p_{n+1}(x) + p_{n+1}(x)^2] \quad (4.18.42)$$

for $n = 1, 2, \dots$

Identity (4.18.42) can easily be proved by induction in the same way in which the Christoffel–Darboux formula (3.13) is usually verified (cf. [Sz2, p. 43]). As I wrote in [Ne36], I think of Theorem 4.18.12 as a rare gem whose significance is hard to overestimate (cf. [DoNe]) and which I predict to become fundamental in future research on spectral properties of Jacobi matrices and the self-adjoint operators they represent.

The favorite result I referred to is the following.

THEOREM 4.18.13 [Ne35]. *Let w be a Freud weight defined by (4.18.1), where m is an even positive integer. Then for every $0 < c < 1$ there exists a constant $c_1 = c_1(c)$ such that*

$$w(x) p_n(x)^2 \leq c_1 n^{-1/m} \quad (4.18.43)$$

for $n = 1, 2, \dots$ and $|x| \leq 2c[r(2^{-1}m)r(2^{-1}m+1)r(m+1)^{-1}]^{1/m} n^{1/m}$. Moreover, there exist three positive constants c_2, c_3 and c_4 such that

$$w(x_{kn}) p_{n-1}(x_{kn})^2 \leq c_2 n^{-1/m}, \quad k = 1, 2, \dots, n, \quad (4.18.44)$$

and

$$w(x_{kn}) p_{n-1}(x_{kn})^2 \geq c_3 n^{-1/m}, \quad |x_{kn}| \leq c_4 n^{1/m} \quad (4.18.45)$$

for $n = 1, 2, \dots$.

Sketch of Proof of Theorem 4.18.13. By (4.18.41)–(4.18.42), we have

$$\sum_{k=0}^{n-1} |a_{k+1}^2 - a_k^2| p_k(x)^2 \geq a_n^2 [1 - x^2/(4a_n^2)] p_n(x)^2 \quad (4.18.46)$$

for $|x| \leq 2a_n \sim 2c[r(2^{-1}m)r(2^{-1}m+1)r(m+1)^{-1}]^{1/2} n^{1/m}$ (cf. Theorem 4.18.3), and by Theorem 4.18.5, one can obtain asymptotics for $a_{k+1}^2 - a_k^2$ which enables one to estimate the right-hand side of (4.18.46) in terms of the reciprocal of the Christoffel function $\lambda_n(w, x)$ for which Theorem 4.16.6 provides upper bounds. This leads to the proof of (4.18.43), and the remaining two inequalities are proved in a similar fashion. ■

This theorem is an improvement of some inequalities of S. S. Bonan [Bon2], and it has recently been generalized by Lubinsky [Lu5]. In [BonCl] there are a number of most interesting inequalities which among other things show that inequality (4.18.43) is no longer valid with $c = 1$. Although there are no proofs in [BonCl], I am familiar with the contents of the draft of the follow-up paper with complete proofs, and I have good reason to believe that the proof of the following result is correct.

THEOREM 4.18.14 [BonCl]. *Let $m > 0$ be even, and let w be a Freud weight defined by (4.18.1). Then*

$$\max_{x \in \mathbb{R}} w(x) p_n(x)^2 \sim n^{1/3 - 1/m} \quad (4.18.47)$$

for $n = 1, 2, \dots$.

The estimate (4.18.47) disproves a conjecture which I made in [Ne17]. Thus I am in an urgent need to make another

Conjecture 4.18.15. Let $d\alpha$ be a measure on the real line, and assume that there exists $m > 1$ such that

$$\lim_{x \rightarrow \infty} |x|^{-m} \log \alpha'(x) = -1. \quad (4.18.48)$$

Then the recurrence coefficients a_n and b_n satisfy

$$\lim_{n \rightarrow \infty} n^{-1/m} a_n = [\Gamma(2^{-1}m) \Gamma(2^{-1}m + 1) \Gamma(m + 1)^{-1}]^{1/m} \quad (4.18.49)$$

and

$$\lim_{n \rightarrow \infty} n^{-1/m} b_n = 0. \quad (4.18.50)$$

I want to bring the reader's attention to Freud's [Fr49, Fr56, Fr59, FrNe2], Lubinsky and A. Sharif's [LuSh], Saff's [Sa], and my [Ne9, Ne11, Ne36] as references for further orientation regarding Freud's conjectures.

4.19. *Quadrature Sums and Lagrange Interpolation Revisited*

For me it is not arguable that Freud was one of the initiators of research on Gauss–Jacobi quadrature processes and Lagrange interpolation on infinite intervals. His two papers [Fr19, Fr33] contribute considerably towards breaking the ice. The former deals with $L_2(w)$ convergence of Lagrange interpolation, which is equivalent to the convergence of the corresponding Gauss–Jacobi quadrature sums, whereas the latter discusses pointwise convergence of Lagrange interpolation taken at the zeros of Hermite polynomials. Freud's [Fr19] is best classified as a response to and improvement of J. Balázs and P. Turán's [BalTu], which both Freud and Turán believed to be the first paper on $L_2(w)$ convergence of Lagrange interpolation on the whole real line. However, unknown to both of them, in 1928 J. V. Uspensky [Us] published a paper in *Transactions of the American Mathematical Society* dealing with essentially the same problem as [BalTu, Fr19].

Knowing Freud's deeply rooted interest in interpolation, quadrature processes and other approximations on infinite intervals, it is hard if not impossible for me to understand why, after his initial achievements, Freud stopped just behind the doorstep and subsequently did not attempt to obtain more than just routine results (cf. [Fr26] for a standard Lebesgue function estimate).

As a result of his neglect of the subject, there are only two areas which have been well researched. One is weighted mean convergence of Lagrange interpolation at the zeros of classical orthogonal polynomials (Hermite and Laguerre), which I started in [Ne17, Ne26] and which was further investigated by Bonan [Bon1] (cf. [KnLu1] for generalizations to Freud weights) and applied by the Australian school of numerical analysis [SmiSIop] to product integration rules. The other topic is related to convergence of Gauss–Jacobi quadratures associated with Freud-type weights

for function majorized by certain entire functions, which was developed by Lubinsky in [Lu1, Lu3, Lu6] and (with A. Sidi) [LuSi2].

There are two inherent problems associated with infinite intervals. First, polynomials are not only not dense in the space of continuous functions, but even $L_p(w)$ density is dependent upon the uniqueness of the solution of the moment (cf. [Fr31b, Chap. II; ShoTa]). Second, we know so little about orthogonal polynomials on infinite intervals that frequently we are faced with genuine difficulties, some of which would be trivial to resolve if w were supported in a finite interval. As a result we are only at an initial stage of developing a theory of Gauss–Jacobi quadratures and Lagrange interpolation in \mathbb{R} .

In Sections 4.7 and 4.8 I discussed the significance of inequalities such as

$$\sum_{k=1}^n |\Pi(x_{kn})|^p \lambda_{kn} \leq K \int_{\mathbb{R}} |\Pi(x)|^p w(x) dx, \quad (4.19.1)$$

where x_{kn} and λ_{kn} denote the zeros and Cotes numbers, respectively (cf. (3.5) and (3.6)). Well, for infinite intervals we do not have sharp uniform estimates for all Cotes numbers, even in the simplest case of Hermite weights where $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$. The problem is that we cannot handle the Cotes numbers γ_{kn} for k near 1 and n (cf. (4.16.5)). Following the methods described in Section 4.7, I was able to circumvent this problem by demonstrating the following theorem in [Ne26, Lemma 5, p. 265] (cf. [Ne17, p. 191]), which turned out to be sufficient for proving weighted $L_p(w)$ convergence of Lagrange interpolation at zeros of Hermite polynomials.

THEOREM 4.19.1 [Ne26]. *Let $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$, and let $0 < c < 1/2$ be fixed. Let $1 \leq p < \infty$, $a \in \mathbb{R}$, $0 \leq b < 1$ and $m \in \mathbb{N}$. Then there exists a positive constant $K = K(w, c, p, a, b, m)$ such that*

$$\begin{aligned} & \sum_{k=[cn]+1}^{[(1-c)n]} |\Pi(x_{kn})|^p w(x_{kn})^{-b} (1 + |x_{kn}|)^a \lambda_{kn} \\ & \leq K [1 + (m/n)^{1/2}] \int_{\mathbb{R}} |\Pi(x)|^p w(x)^{1-b} (1 + |x|)^a dx \end{aligned} \quad (4.19.2)$$

for all polynomials Π of degree at most m .

Strictly speaking, I proved (4.19.2) only for $p = 1$, $a = -1$ and $b = 1/2$ in [Ne26], but the general case is proved in the same way, using Bernstein–Markov inequalities (cf. Section 4.17), Christoffel function estimates (cf. Section 4.16) and Markov–Stieltjes inequalities (cf. [Fr31b, Sect. 1.5]). The following would be useful indeed and is probably true, although I have no idea as to the method of proof.

Conjecture 4.19.2. Theorem 4.19.1 remains valid if $c = 0$ in (4.19.2).

What I can prove at the present time is that inequality (4.19.2) holds with $c = 0$ if a is replaced by $a + 2$ on the right-hand side.

In a recent paper Lubinsky, Máté and I [LuMáNe] succeeded in generalizing Theorem 4.19.1 to general Freud weights for all $p > 0$ except that, instead of all $0 < c < \frac{1}{2}$, we were able to prove our result only for $0 < c < c^*$ where $0 < c^* = c^*(w) < \frac{1}{2}$. Our proof was based on the same ideas used to prove Theorem 4.7.5, and we could not handle the case of all values of $c \in (0, \frac{1}{2})$ because of the lack of appropriate Christoffel function estimates close to the greatest zeros of orthogonal polynomials with Freud-type weights (cf. Theorem 4.16.5). Related inequalities were also proved by Bonan [Bon1] and Knopfmacher and Lubinsky [KnLu1].

There is another way of estimating quadrature sums in terms of integrals which is based on an inequality of A. A. Markov,

$$\sum_{k=1}^n F(x_{kn}) \lambda_{kn} \leq \int_{\mathbb{R}} F(x) d\alpha(x) \quad (4.19.3)$$

(cf. [Marko, p. 81; Sz2, Exercise 9, p. 378]), which is true for all measures $d\alpha$ provided F satisfies $F^{(2j)}(x) \geq 0$ for $x \in \mathbb{R}$ and $j = 1, 2, \dots, n$. I have two interesting remarks in connection with (4.19.3). First, in his book [Fr31b, p. 136], Freud attributes (4.19.3) to Balázs and Turán, who, in [BalTu], innocently rediscovered it (Turán was one of the referees of Freud's [Fr31a, b]). Second, (4.19.3) is given as Lemma 3.1.5 in [Fr31b, p. 92], and I strongly urge the reader to open Freud's book to page 92 and examine carefully the illustration (Fig. 7) accompanying the condition that the even derivatives of F are nonnegative (the graph of the function is not convex!).

There are several applications of (4.19.3) which are crucial when one is trying to prove

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_{kn}) \lambda_{kn} = \int_{\mathbb{R}} f(x) d\alpha(x) \quad (4.19.4)$$

for one or another class of functions f . The point is that, for unbounded functions, the quadrature sums need not be uniformly bounded, even if the corresponding integral is. However, if f is dominated by a function $F \in L_1(d\alpha)$ whose even derivatives are nonnegative, then, by (4.19.3), the associated quadrature sums are always uniformly bounded. On the basis of this observation, one can easily prove the following result of Freud [Fr19, Theorem 3, p. 266] (cf. [Fr31b, p. 93]), which is a generalization of Balázs and Turán's [BalTu, Theorem A, p. 470].

THEOREM 4.19.3 [Fr19]. *Suppose that, for $d\alpha$, the moment problem possesses a unique solution. Let f be bounded on every finite interval and let*

$$\int_{\mathbb{R}} f(x) d\alpha(x) < \infty \quad (4.19.5)$$

exists in the improper Riemann–Stieltjes sense. Assume that there is an infinitely differentiable function $F \in L_1(d\alpha)$ whose even-order derivatives are all nonnegative such that

$$\lim_{x \rightarrow \infty} f(x)/F(x) = 0. \quad (4.19.6)$$

Then the quadrature convergence (4.19.4) takes place.

Hence the name of the game is to find entire functions $F \in L_1(d\alpha)$ with nonnegative Taylor coefficients which grow as fast as possible as $x \rightarrow \infty$. Sometimes this is easy (e.g., w is the Hermite weight); sometimes this is somewhat complicated, requiring sophisticated arguments (e.g., w is a Freud weight such as $w(x) = \exp(-|x|^m)$, $m > 0$, or $w(x) = \exp(-Q(x))$, where Q satisfies certain conditions similar to those formulated in Theorem 4.16.4). Lubinsky's [Lu3, formula (17)] function F , defined by

$$F(x) = 1 + \sum_{n=1}^{\infty} (cx/q_n)^{2n} n^{-1/2} w(q_n)^{-1} \quad (4.19.7)$$

($0 < c < 1$), does have this property where q_n is the unique positive solution of the equation $qQ'(q) = n$. In Lubinsky and Sidi's paper [LuSi2], a variety of related results are proved on convergence of product integration rules formed from Gauss–Jacobi quadratures.

Another application of (4.19.3) yields estimates for

$$\sum_{k=1}^n w(x_{kn})^{-1} \lambda_{kn}, \quad (4.19.8)$$

which is a quadrature sum for a divergent integral $\int w/w$. Sums such as (4.19.8) come up naturally when one is investigating quadrature sums and Lagrange interpolating polynomials for a function whose growth one would want to control with the least restrictive conditions. One expects (4.19.8) to behave like

$$\int_{x_{nn}}^{x_{1n}} [w(x)/w(x)] dx = (x_{1n} - x_{nn}) \quad (4.19.9)$$

since the quadrature sum (4.19.8) is not affected by the values of w taken outside the interval $(x_{1n} - x_{nn})$. It turns out that this argument can indeed

be justified for certain weight functions. Using ideas from [Ne8, Lemma, p. 89] and a beautiful generalization of (4.19.3) to a Markov–Stieltjes-type inequality, Knopfmacher and Lubinsky [KnLu1, Theorem 6] proved the following

THEOREM 4.19.4 [KnLu1]. *Let w be defined by*

$$w(x) = \exp(-Q(x)), \quad x \in \mathbb{R}, \quad (4.19.10)$$

where Q is even, nonnegative, and increasing for $x > 0$; Q'' is nondecreasing in (c_1, ∞) ; and

$$0 < xQ''(x)/Q'(x) \leq c_2 \quad (4.19.11)$$

with some positive constants c_1 and c_2 . Then

$$\sum_{k=1}^n w(x_{kn})^{-1} \lambda_{kn} \sim q_n, \quad n = 1, 2, \dots, \quad (4.19.12)$$

where q_n is the unique positive solution of $qQ'(q) = n$.

Weighted L_p convergence of Lagrange interpolation is a serious business requiring delicate analysis of several aspects of orthogonal polynomials to such an extent that at the present time it is only wishful thinking that it has been completely resolved for Freud-type weights. The ability to produce sharp pointwise estimates for the orthogonal polynomials seems to be a necessary ingredient for weighted $L_p(w)$ convergence and this has been accomplished only for w given by

$$w(x) = \exp(-x^m), \quad x \in \mathbb{R}, \quad (4.19.13)$$

where m is an even positive integer (cf. Theorems 4.18.13 and 4.18.14).

There are two sides to this issue. One concerns necessary conditions, whereas the other involves sufficient conditions for weighted L_p convergence of Lagrange interpolation. For measures supported in $[-1, 1]$, Theorem 4.8.2 gives a more than satisfactory solution of the former problem. On infinite intervals, my proof will still work provided one knows the behavior of the Cotes numbers, recurrence coefficients, and zeros associated with the orthogonal polynomials and is able to control the two quantities

$$\sum_{k \in I} |P_{n-1}(w, x_{kn})|^p \lambda_{kn} \quad (4.19.14)$$

(I is a certain set of indices k) and

$$\int_{\mathbb{R}} |p_n(w, t)|^p u(t) dt \quad (4.19.15)$$

($u \geq 0$) by proving sharp two-sided estimates for them.

For Hermite polynomials, I found necessary conditions for weighted $L_p(w)$ convergence of Lagrange interpolation in [Ne26, Theorem 2, p. 265], and my results were extended to the generalized Hermite weight

$$w(x) = |x|^a \exp(-x^2), \quad x \in \mathbb{R}, \quad a > -1 \quad (4.19.16)$$

(cf. [Sz2, Exercise 25, p. 380]), by Bonan in [Bon1]. By a quadratic transformation, the weight in (4.19.16) becomes the Laguerre weight function, and thus Bonan's results do include the case of Lagrange interpolation at zeros of Laguerre polynomials. In [Ne39] I proved the following

THEOREM 4.19.5 [Ne39]. *Let w be defined by (4.19.13) where $m > 0$ is even. Let $u (\geq 0) \in L_1[\mathbb{R}]$ and $0 < p < \infty$ be given. Let*

$$\int_{\mathbb{R}} [w(t)^{1/2} (1 + |t|)]^{-p} u(t) dt = \infty. \quad (4.19.17)$$

Then there exists a function f supported in a finite interval such that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}} |L_n(w, f, t)|^p u(t) dt = \infty. \quad (4.19.18)$$

One interesting feature of the proof of Theorem 4.19.5 is that it uses Dombrowski's lovely Theorem 4.18.12 to show that $a_n p_n(w, x)^2$ and $a_{n+1} p_{n+1}(w, x)^2$ cannot be small at the same time, i.e.,

$$a_n p_n(w, x)^2 + a_{n+1} p_{n+1}(w, x)^2 \geq K/w(x), \quad |x| \leq cn^{1/m}, \quad (4.19.19)$$

with suitable positive constants K and c .

Choosing $u = w^{p/2}$, the integral in (4.19.17) becomes convergent for all $1 < p < \infty$, and thus Theorem 4.19.5 suggests that one may hope for weighted L_p convergence of Lagrange interpolation in this case. This is indeed true, at least for w given by (4.19.13) and (4.19.16). Such problems were investigated in [Ne17, Ne26, Bon1, KnLu1] (in chronological order) although I hesitate to quote the most general results proved by Knopfmacher and Lubinsky given the preliminary state of their manuscript [KnLu1], which, in view of [BonCl], is due for a reevaluation and/or revision. (In all fairness, I must point out that within two months after I read one of the first drafts of this work, Knopfmacher and Lubinsky

prepared a revised version of [KnLu1].) Contrary to the case of finite intervals where Askey's method of reducing weighted L_p convergence of Lagrange interpolation to that of orthogonal Fourier series is the main tool of the trade (cf. Theorem 4.8.4), here we cannot rely on orthogonal Fourier series since they converge only for an excessively limited range of the parameter p (cf. [AsWa1, Mu2, Mu3]). Instead, one needs to generalize the method used to prove Theorem 4.8.6.

For Lagrange interpolation at the zeros of Hermite polynomials, I proved the following result in [Ne26, Theorem 1, p. 264] (cf. [Ne17, Theorem 16, p. 190]) in spite of Askey's pessimistic predictions made in [As4, p. 84] which I quoted at the end of Section 4.8.

THEOREM 4.19.6 [Ne26]. *Let $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$. Let f be a continuous function on \mathbb{R} satisfying*

$$\lim_{|x| \rightarrow \infty} f(x)(1 + |x|) \exp(-x^2/2) = 0. \quad (4.19.20)$$

Then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} [|f(t) - L_n(w, f, t)| \exp(-t^2/2)]^p dt = 0 \quad (4.19.21)$$

for every $1 < p < \infty$.

Weighted L_p convergence results such as Theorem 4.19.6 easily yield convergence of product integration rules which are defined as follows.

Let $d\alpha$ be a measure on \mathbb{R} , and let W be a Lebesgue-measurable function on \mathbb{R} such that all the moments of $|W|$ with respect to the Lebesgue measure are finite. The product integration rule $I_n(d\alpha, W)$ based on the zeros $\{x_{kn}\}$ of $p_n(d\alpha, x)$ is

$$I_n(d\alpha, W, f) = \sum_{k=1}^n f(x_{kn}) A_{kn}, \quad (4.19.22)$$

where the weights A_{kn} are chosen so that

$$I_n(d\alpha, W, P) = \int_{\mathbb{R}} P(x) W(x) dx \quad (4.19.23)$$

for every polynomial P of degree at most $n - 1$.

Convergence of product integration rules is, in a sense, equivalent to weighted L_p convergence of Lagrange interpolation, a fact first understood by I. H. Sloan and W. E. Smith [SlSmi2] and subsequently used in papers such as [SmiSl, SmiSlOp, KnLu1]. What all these researchers failed to point out is that more general product integration rules for zeros of quasi-

orthogonal polynomials were introduced previously in Freud's [Fr31b, Problems 1, 2, 3, 5, pp. 130–131], where the formula

$$A_{kn} = \lambda_{kn}(d\alpha) S_n(d\alpha, W, x_{kn}) \quad (4.19.24)$$

is given as Problem 1. Here $S_n(d\alpha, W)$ is the partial sum of the orthogonal Fourier series of W (cf. (3.10)). Not being a numerical analyst, or for that matter, a practical-minded person, Freud did not pursue this matter further, and thus it remained hidden from applied mathematicians, who are less interested in the theoretical aspects of such integration rules.

Formula (4.19.24) explains the delicate connection between product integration rules on the one hand and Lagrange interpolation and orthogonal Fourier series on the other.

There have been no new developments regarding pointwise convergence of Lagrange interpolation taken at zeros of orthogonal polynomials associated with Hermite, Laguerre or possibly Freud-type weights in the past 10 years, and what had previously been known was the subject of my survey paper [Ne17, pp. 168–176], including Freud's [Fr33], which is more remarkable for its Christoffel function estimates than for its results on convergence of Lagrange interpolation, although the latter are fairly interesting as well.

My favorite result on pointwise convergence of Lagrange interpolation is concerned with the Dini–Lipschitz condition, or more accurately, with its one-sided generalization which I introduced in [Ne12] (cf. [Ne5, Ne6, Ne14, Ne15, Ne17]). The following proposition was proved in [Ne15, Theorem 5, p. 345].

THEOREM 4.19.7 [Ne15]. *Let $w(x) = \exp(-x^2)$, $x \in \mathbb{R}$. Let f be an almost everywhere continuous function on \mathbb{R} satisfying*

$$f(x) \leq K \exp(-cx^2), \quad x \in \mathbb{R}, \quad (4.19.25)$$

where $K > 0$ and $c < \frac{1}{2}$. Assume that, on an interval $[a, b]$, the function f satisfies the one-sided Dini–Lipschitz condition

$$f(x+t) - f(x) \geq -v(t) |\log t|^{-1}, \quad a < x < x+t < b, \quad (4.19.26)$$

where $v(t) \searrow 0$ for $t \rightarrow +0$. Then

$$\lim_{n \rightarrow \infty} L_n(w, f, x) = f(x) \quad (4.19.27)$$

if f is continuous at $x \in (a, b)$, and the convergence is uniform in every closed subinterval of (a, b) where f is continuous.

The point is that increasing functions automatically satisfy the one-sided Dini–Lipschitz condition (4.19.26), and thus (4.19.27) holds for functions of bounded variation as well.

4.20. *Differential Equations and Freud Polynomials*

On the basis of some ideas originating with E. N. Laguerre [La], J. A. Shohat [Sho3, Sho7] showed that the reason that some orthogonal polynomials satisfy differential equations is to be found in the intrinsic nature of the weight function itself, and the weight function is not just a passive carrier of the genes but, in fact, these properties can be recovered by a very clever argument. There are a number of characterizations of classical orthogonal polynomials in terms of the differential equations satisfied either by these polynomials or by the associated weight functions (cf. [Ac, Cs, Hah, Kr1, Kr2]). Shohat’s method, however, enables one to investigate other orthogonal polynomials as well, including a special class of Freud polynomials where the weight function w can be written as

$$w(x) = \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \quad (4.20.1)$$

or in a somewhat more general form

$$w(x) = |x|^a \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \quad (4.20.2)$$

where Π_m is a polynomial of degree m with positive leading coefficient. I rediscovered Shohat’s method in [Ne29], where I carried out a systematic study of orthogonal polynomials associated with $\exp(-x^4)$ which I continued in [Ne31]. The fever caught other devotees, and at the present time I can make reasonably accurate predictions of the nature of orthogonal polynomials associated with weights (4.20.1) or (4.20.2).

What distinguishes w in (4.20.1) from any other weight is that it satisfies the differential equation

$$w' = R w, \quad R \in \mathbb{P}_{m-1} \quad (4.20.3)$$

($R = -\Pi'_m$).

The simplest case, when w is the Hermite weight function, that is,

$$w(x) = \exp(-x^2), \quad x \in \mathbb{R}, \quad (4.20.4)$$

leads to the differential equations

$$y'' - 2xy' + 2ny = 0, \quad y = p_n(w, x), \quad (4.20.5)$$

and

$$z'' + (2n + 1 - x^2)z = 0, \quad z = w(x)^{1/2} p_n(w, x). \quad (4.20.6)$$

Before explaining Shohat's technique of finding differential equations, I will demonstrate his method on the example of Hermite polynomials which I will temporarily denote by $h_n = p_n(w)$. Since the Hermite weight is even, the recurrence formula (3.7) for the Hermite polynomials takes the form

$$xh_n(x) = a_{n+1}h_{n+1}(x) + a_n h_{n-1}(x), \quad (4.20.7)$$

where $h_n(x) = \gamma_n x^n + \dots$, $a_n = \gamma_{n-1}/\gamma_n$. By orthogonality

$$n/a_n = \int_{\mathbb{R}} [h_n(x) h_{n-1}(x)]' \exp(-x^2) dx, \quad (4.20.8)$$

and thus on the basis of (4.20.3) ($R(x) = -2x$), integration by parts yields

$$n/a_n = 2 \int_{\mathbb{R}} h_n(x) h_{n-1}(x) x \exp(-x^2) dx = 2a_n \quad (4.20.9)$$

so that

$$a_n = (n/2)^{1/2}. \quad (4.20.10)$$

If Q is an arbitrary polynomial of degree less than $n-1$, then, again by orthogonality relations and (4.20.3), we obtain

$$\begin{aligned} \int_{\mathbb{R}} h'_n(x) Q(x) \exp(-x^2) dx &= \int_{\mathbb{R}} [h_n(x) Q(x)]' \exp(-x^2) dx \\ &= 2 \int_{\mathbb{R}} h_n(x) Q(x) x \exp(-x^2) dx = 0. \end{aligned} \quad (4.20.11)$$

Therefore h'_n is the orthogonal to all polynomials of degree at most $n-2$, that is,

$$h'_n(x) = \text{const } h_{n-1}(x) \quad (4.20.12)$$

and comparison of leading coefficients yields

$$h'_n(x) = (2n)^{1/2} h_{n-1}(x). \quad (4.20.13)$$

Differentiating (4.20.13), we obtain

$$h''_n(x) = 2[n(n-1)]^{1/2} h_{n-2}(x), \quad (4.20.14)$$

and applying the recurrence formula (4.20.7) (cf. (4.20.10)) to h_{n-2} , we end up with

$$h''_n(x) = 2(2n)^{1/2} (xh_n(x) - (n/2)^{1/2} h_{n-1}(x)). \quad (4.20.15)$$

Now we can eliminate h_{n-1} from (4.20.13) and (4.20.15) to obtain the differential equation (4.20.5), from which (4.20.6) follows as well.

Let us make the ideas that lead to the differential equation of Hermite polynomials crystal clear. First, the recurrence coefficient in (4.20.7) is evaluated, and this is done via application of (4.20.3). Then the Luzinian (4.20.13) is proved, and again property (4.20.3) is used in the proof. Finally, (4.20.13) combined with the recurrence formula (4.20.7) immediately yields the differential equation (4.20.5).

I call (4.20.13) Luzinian because it was N. N. Luzin [Luz, p. 50] who asked whether there are any orthogonal systems in addition to the trigonometric system that are invariant under either differentiation or integration. In view of results of B. M. Gaganov [Ga1–Ga3], Ya. L. Geronimus [Ger1], W. Hahn [Hah], H. L. Krall [Kr1, Kr2], and others, we know that the Hermite polynomials (modulo a linear transformation of the variable) are the only orthogonal polynomials that are invariant under differentiation. At the same time (4.20.13) seems to be of crucial significance in establishing the differential equation. What Shohat realized and what was independently discovered approximately 40 years later by Freud and his school (including S. S. Bonan, H. N. Mhaskar, and me) is that derivatives of orthogonal polynomials associated with weights w representable as (4.20.1) are quasi-orthogonal, and the notion of quasi-orthogonality is a perfect substitute for orthogonality. (I personally feel somewhat guilty of ignorance in this case; had I been familiar with Shohat's work, I would have been able to do much more and much earlier than I actually did.)

For a given orthogonal polynomial system $\{p_n(d\alpha)\}$, the derivative system $\{p'_n(d\alpha)\}$ is called quasi-orthogonal (of order m) if there is an integer m ($m \geq 2$) such that, for all n ,

$$p'_n(d\alpha, x) = \sum_{k=n-m+1}^{n-1} c_{kn} p_k(d\alpha, x), \quad (4.20.16)$$

where the coefficients c_{kn} may of course depend on the measure $d\alpha$.

It is an easy exercise to prove that the orthogonal polynomials $p_n(w, x)$ associated with the weight w in (4.20.1) are quasi-orthogonal of order precisely m . To see this, we pick an arbitrary polynomial Q of degree at most $n - m$. Then, by (4.20.3),

$$\begin{aligned} \int_{\mathbb{R}} p'_n(w, x) Q(x) w(x) dx &= \int_{\mathbb{R}} [p_n(w, x) Q(x)]' w(x) dx \\ &= - \int_{\mathbb{R}} p_n(w, x) Q(x) P'_m(x) w(x) dx = 0 \end{aligned} \quad (4.20.17)$$

and thus (4.20.16) holds.

In [BonNe] we characterized all orthogonal polynomials whose derivatives are quasi-orthogonal of orders three and four as follows.

THEOREM 4.20.1 [BonNe]. *Let $\{p_n(d\alpha)\}$, $n=0, 1, \dots$, be a system of orthonormal polynomials corresponding to some measure $d\alpha$. Then the following statements are equivalent.*

(i) *There exist two integers j and k and two sequences $\{e_n\}$ and $\{c_n\}$, $n=1, 2, \dots$, such that $j < k$ and*

$$p'_n(d\alpha, x) = e_n p_{n-j}(d\alpha, x) + c_n p_{n-k}(d\alpha, x) \quad (4.20.18)$$

for $n=1, 2, \dots$

(ii) *There exists a nonnegative constant c such that*

$$p'_n(d\alpha, x) = n/a_n p_{n-1}(d\alpha, x) + ca_n a_{n-1} a_{n-2} p_{n-3}(d\alpha, x) \quad (4.20.19)$$

for $n=1, 2, \dots$, where a_n denotes the recursion coefficient in (3.7).

(iii) *There exist three real numbers c , b and K such that $c \geq 0$, if $c = 0$ then $K > 0$, and the recursion coefficients a_n and b_n in (3.7) satisfy*

$$n = ca_n^2 [a_{n+1}^2 + a_n^2 + a_{n-1}^2] + Ka_n^2 \quad (4.20.20)$$

for $n=1, 2, \dots$ and

$$b_n = b \quad (4.20.21)$$

for $n=0, 1, \dots$

(iv) *The measure $d\alpha$ is absolutely continuous and there exist four real numbers D , c , b and K such that $D > 0$, $c \geq 0$, if $c = 0$ then $K > 0$, and*

$$\alpha'(x) = D \exp[-c(x-b)^4/4 - K(x-b)^2/2], \quad x \in \mathbb{R}. \quad (4.20.22)$$

Regarding the different constants in (4.20.18)–(4.20.22), we can say the following. If c is given by one of the statements (ii), (iii) or (iv), then in the remaining statements it has the same value. The same comment applies to b and K in (iii) and (iv). If c is given by (ii), then b and K in (iii) and (iv) would still be arbitrary except that, if $c = 0$, then K must be positive.

Complete characterization of orthogonal polynomials with quasi-orthogonal derivatives is given in [BonLuNe], where, among other results, we prove

THEOREM 4.20.2 [BonLuNe]. *The derivatives of an orthogonal polynomial system $\{p_n(d\alpha)\}$, $n=0, 1, \dots$, are quasi-orthogonal of order m in*

the sense of (4.20.16) if and only if the measure dx is absolutely continuous and

$$\alpha'(x) = \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \quad (4.20.23)$$

where $\Pi_m \in \mathbb{P}_m$.

Note that a somewhat weaker result was proved in [HeRo] (cf. [Ro1, VanR1, VanR2]).

Now I can proceed with describing

SHOHAT'S METHOD. Assume that w is defined by (4.20.1). Then according to Theorem 4.20.2, one can expand $p'_n(w)$ as given in (4.20.16). Since the orthogonal polynomials satisfy the three-term recurrence (3.7), repeated application of (3.7) leads to

$$p_k(w, x) = A_{kmn}(x) p_n(w, x) + B_{kmn}(x) p_{n-1}(w, x), \quad (4.20.24)$$

$n - m + 1 \leq k \leq n - 2$, where A_{kmn} and B_{kmn} are polynomials of degree at most $m - 1$ with coefficients depending on k, m, n and the recursion coefficients in (3.7). Substituting (4.20.24) in (4.20.16) yields

$$p'_n(w, x) = A_{mn}(x) p_n(w, x) + B_{mn}(x) p_{n-1}(w, x), \quad (4.20.25)$$

where A_{mn} and B_{mn} are polynomials of degree at most $m - 1$ with coefficients depending on m, n and the recursion coefficients in (3.7). Differentiating (4.20.25), we obtain

$$\begin{aligned} p''_n(w, x) &= A'_{mn}(x) p_n(w, x) + B'_{mn}(x) p_{n-1}(w, x) \\ &\quad + A_{mn}(x) p'_n(w, x) + B_{mn}(x) p'_{n-1}(w, x), \end{aligned} \quad (4.20.26)$$

where we can apply (4.20.25) and the recursion formula (3.7) to express $p'_n(w)$ and $p'_{n-1}(w)$ in terms of $p_n(w)$ and $p_{n-1}(w)$. Proceeding in this way, we can rewrite (4.20.26) as

$$p''_n(w, x) = C_{mn}(x) p_n(w, x) + D_{mn}(x) p_{n-1}(w, x), \quad (4.20.27)$$

where C_{mn} and D_{mn} are again polynomials of degree at most m with coefficients depending on m, n and the recursion coefficients in (3.7). Now we can eliminate $p_{n-1}(w)$ from (4.20.25) and (4.20.27). What we get is Shohat's

THEOREM 4.20.3 [Sho7]. *If w is given by (4.20.1), then the corresponding orthogonal polynomials satisfy the second-order linear homogeneous differential equation*

$$E_{mn} y'' + F_{mn} y' + G_{mn} y = 0, \quad y = p_n(w, x), \quad (4.20.28)$$

where E_{mn} , F_{mn} and G_{mn} are polynomials of degree at most m , m and $2m$, respectively, with coefficients depending on m , n and the recursion coefficients in (3.7).

So far there have been only a few cases where the differential equation has been determined explicitly; these include $w(x) = \exp(-x^4)$ by Shohat [Sho7] and me [Ne29, Theorem 10, p. 277], $w(x) = \exp(-x^6)$ by R. C. Sheen [Sh1, Sh2], and $w(x) = \text{expexp}(-\Pi_4(x))$ by W. C. Bauldry [Bau2, Theorem 3.3.3, p. 67]. For instance, Bauldry's differential equation is as follows.

THEOREM 4.20.4 [Bau2]. *Let*

$$w(x) = \exp(-\Pi_4(x)), \quad x \in \mathbb{R}, \quad (4.20.29)$$

where

$$\Pi_4(x) = x^4/4 + q_3 x^3/3 + q_2 x^2/2 + q_1 x, \quad (4.20.30)$$

$q_1, q_2, q_3 \in \mathbb{R}$. Let φ_n and ψ_n be defined by

$$\varphi_n(x) = a_{n+1}^2 + a_n^2 + b_n^2 + b_n q_3 + q_2 + x^2 + x b_n + x q_3 \quad (4.20.31)$$

and

$$\psi_n(x) = b_{n-1} + b_n + q_3 + x, \quad (4.20.32)$$

where a_n and b_n are the recursion coefficients in (3.7) satisfying equations analogous to (4.18.13). Then the function z given by

$$z(x) = p_n(w, x)[w(x)/\varphi_n(x)]^{1/2} \quad (4.20.33)$$

satisfies

$$z'' + \left\{ -3(\varphi_n'/\varphi_n)^2/4 - (\varphi_n'/\varphi_n) \Pi_4/2 - (\Pi_4/2)^2 + (\varphi_n''/\varphi_n)/2 + \Pi_4''/2 + a_n^2[1 + \varphi_{n-1}\varphi_n - \psi_n(\Pi_4' + \varphi_n'/\varphi_n + a_n^2\psi_n)] \right\} z = 0. \quad (4.20.34)$$

Naturally, if some of the parameters in (4.20.30) vanish, then (4.20.34) somewhat simplifies. According to the announcement [BonC1], the equation for $w(x) = \exp(-x_m)$, m even, is very similar to (4.20.34) although one will probably be unable to express the coefficients of the general equation explicitly in terms of the recursive coefficients in (3.7). Instead, suitable asymptotic equations will be found which will be sufficient for finding asymptotic properties of the corresponding orthogonal polynomials.

In conclusion, I point out that Shohat's method of constructing differential equations for orthogonal polynomials is applicable to a variety of weight functions and/or measures. For instance, if w is given by (4.20.2), that is, the weight has an algebraic singularity at 0, then (4.20.16) can be replaced by

$$xp'_n(d\alpha, x) = \sum_{k=n-m+1}^n c_{kn} p_k(d\alpha, x) \quad (4.20.35)$$

and then the above arguments leading to Shohat's Theorem 4.20.3 can be repeated to obtain a differential equation. In [BonLuNe], S. S. Bonan, D. S. Lubinsky, and I introduced a generalized notion of quasi-orthogonality which amounts to the possibility of writing

$$Q_s(x) p_n^{(j)}(d\alpha, x) = \sum_{k=n-m+1}^{n-j+s} c_{kn} p_k(d\alpha, x) \quad (4.20.36)$$

for some positive integer j with a suitable polynomial Q of degree s . We find an exhaustive characterization of all such measures which turn out to be not necessarily absolutely continuous. Clearly, all classical orthogonal polynomials such as Jacobi, Hermite and Laguerre polynomials satisfy (4.20.36). The class of all orthogonal polynomials admitting (4.20.36) is of much greater proportions than just the collection of classical orthogonal polynomials, and all of them possess a reasonably acceptable differential equation of the form (4.20.28). Al. Magnus pointed out to me that the associated classical orthogonal polynomials belong to the Laguerre-Hahn class whose elements satisfy a fourth-order linear differential equation.

4.21. *Plancherel-Rotach Asymptotics for Orthogonal Polynomials with Freud Weights*

The rules of the game are simple: you give me (i) a differential equation such as (4.20.6) (satisfied by the Hermite polynomials), and (ii) an asymptotic expression for the solution of the equation at one point, say, the origin; in turn, it is my task to (iii) devise and prove asymptotics for the solution in an interval as large as possible. Let me elaborate on this.

Step (i). The Differential Equation

If w is given by

$$w(x) = \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \quad (4.21.1)$$

where Π_m is a polynomial of degree m with a positive leading coefficient, then by Shohat's Theorem 4.20.3, the corresponding orthogonal polynomials satisfy

$$E_{mn} y'' + F_{mn} y' + G_{mn} y = 0, \quad y = p_n(w, x), \quad (4.21.2)$$

where E_{mn} , F_{mn} and G_{mn} are polynomials of degree at most m , m and $2m$, respectively, with coefficients depending on m , n and the recursion coefficients in (3.7). Therefore, if E_{mn} , F_{mn} and G_{mn} are known explicitly and one is able to represent the recursion coefficients a_n and b_n by sufficiently accurate asymptotic expressions such as (4.18.16), then the usually monstrous equation (4.21.2) turns into the socially and mathematically more acceptable

$$\text{SET}(y'', y', y) = O(\text{small terms}) y, \quad y = p_n(w, x), \quad (4.21.3)$$

where SET = "Simple Expression in Terms of." Moreover, this can be coupled with standard techniques of eliminating first derivatives by introducing a new function z in (4.21.2) defined by

$$z(x) = y(x) \exp \left\{ - \int F_{mn}(t) / [2E_{mn}(t)] dt \right\} \quad (4.21.4)$$

(cf. [Sz2, p. 16]) to arrive at

$$\text{SET}(z'', z) = O(\text{small terms}) z. \quad (4.21.5)$$

If one is sufficiently lucky (or rather, if there is justice in our universe), then (4.21.5) can be transformed via simultaneously introducing new functions and variables to

$$\text{VSET}(v'', v) = O(\text{small terms}) v, \quad (4.21.6)$$

where VSET = "Very Simple Expression in Terms of." The point is that the homogeneous version of (4.21.6) is supposed to be solvable in terms of elementary functions.

In addition to Hermite polynomials, there are only a few other cases that have been treated according to the plan presented above. Their number is steadily increasing, however, and we are actually at the threshold of a breakthrough which will enable us to handle the general equation associated with w in (4.21.1) (cf. [BonCl]).

The first example I give is the differential equation for the orthogonal polynomials associated with w defined by

$$w(x) = \exp(-x^4), \quad x \in \mathbb{R}. \quad (4.21.7)$$

In this case, (4.21.2) takes the form

$$\begin{aligned} \varphi_n y'' - (4x^3 \varphi_n + 2x) y' \\ + 4a_n^2 (4\varphi_n^2 \varphi_{n-1} + \varphi_n - 4a_n^2 x^2 \varphi_n - 4x^4 \varphi_n - 2x^2) y = 0, \end{aligned} \quad (4.21.8)$$

where

$$\varphi_n(x) = a_{n+1}^2 + a_n^2 + x^2 \quad (4.21.9)$$

and a_n is the recursion coefficient in (3.7) which is the solution of the difference equation

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2), \quad a_0^2 = 0, \quad a_1^2 = r(3/4)/r(1/4), \quad (4.21.10)$$

$n = 1, 2, \dots$ (cf. [Ne29, p. 277; Sho7]). With the substitution

$$z(x) = p_n(w, x)[w(x)/\varphi_n(x)]^{1/2}, \quad (4.21.11)$$

Eq. (4.21.8) becomes

$$\begin{aligned} z'' + [4a_n^2(4\varphi_n\varphi_{n-1} + 1 - 4a_n^2x^2 - 4x^4 - 2x^2/\varphi_n) \\ - 4x^6 - 4x^4/\varphi_n - 3x^2/\varphi_n^2 + 6x^2 + 1/\varphi_n] z = 0 \end{aligned} \quad (4.21.12)$$

(cf. [Ne29, Theorem 10, p. 277]). If we set

$$x = (4n/3)^{1/4} \cos \theta, \quad (4.21.13)$$

$$\tau = \int_{\pi/2}^{\theta} [g(t) + (2n)^{-1}] dt \quad (4.21.14)$$

and

$$v(\tau) = z((4n/3)^{1/4} \cos \theta)[g(\theta) + (2n)^{-1}]^{1/2} [\sin \theta]^{-1/2}, \quad (4.21.15)$$

where

$$g(t) = 1 - 2(\cos 2t)/3 - (\cos 4t)/3 \quad (4.21.16)$$

(cf. [Ne31, pp. 1180–1182]), and we apply (4.18.16) to estimate the expression in brackets in (4.21.12) with sufficient accuracy, then (4.21.12) is transformed into

$$v_{\tau\tau} + n^2v = O(1)v \quad (4.21.17)$$

uniformly for θ in (4.21.13) belonging to any fixed closed subinterval of $(0, \pi)$ (cf. [Ne31, formula (22)]).

The second example is the differential equation for orthogonal polynomials corresponding to

$$w(x) = \exp(-x^6/6), \quad x \in \mathbb{R}. \quad (4.21.18)$$

This was worked out in R. C. Sheen's Ph.D. dissertation [Sh1] and will be published in [Sh2]. The analogues of (4.21.9)–(4.21.12) are

$$z'' + f_n z = 0, \quad (4.21.19)$$

where

$$z(x) = p_n(w, x)[w(x)/\varphi_n(x)]^{1/2} \quad (4.21.20)$$

with

$$\begin{aligned} \varphi_n(x) = & a_{n+1}^2(a_{n+2}^2 + a_{n+1}^2 + a_n^2) + a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2) \\ & + x^2(a_{n+1}^2 + a_n^2 + x^2), \end{aligned} \quad (4.21.21)$$

$$\delta_n(x) = xa_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2 + x^2) \quad (4.21.22)$$

and

$$\begin{aligned} f_n(x) = & -x^{10}/4 - x^5\varphi_n(x)^{-1}\varphi_n(x)'/2 + 5x^4/2 - 3[\varphi_n(x)']^2\varphi_n(x)^{-2}/4 \\ & + \varphi_n(x)^{-1}\varphi_n(x)''/2 + a_n^2\varphi_n(x)\varphi_{n-1}(x) + \delta_n(x)' - \delta_n(x)^2 - \delta_n(x)x^5 \\ & - 4\delta_n(x)x^3\varphi_n(x)^{-1} - 2x\delta_n(x)(a_{n+1}^2 + a_n^2). \end{aligned} \quad (4.21.23)$$

In formulas (4.21.21)–(4.21.23), the coefficients a_n (cf. (3.7)) are the unique positive solutions of the Freud-type recurrence

$$\begin{aligned} n = & a_n^2(a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 + 2a_{n+1}^2 a_n^2 + a_{n+1}^2 a_{n-1}^2 + a_n^4 \\ & + 2a_n^2 a_{n-1}^2 + a_{n-1}^4 + a_{n-1}^2 a_{n-2}^2), \quad a_{-1} = 0, a_0 = 0, n = 1, 2, \dots \end{aligned} \quad (4.21.24)$$

With substitutions

$$x = (32n/5)^{1/6} \cos \theta, \quad (4.21.25)$$

$$\tau = \int_{\pi/2}^{\theta} [g(t) + (2n)^{-1}] dt \quad (4.21.26)$$

and

$$v(\tau) = z((32n/5)^{1/6} \cos \theta)[g(\theta) + (2n)^{-1}]^{1/2} [\sin \theta]^{-1/2}, \quad (4.21.27)$$

where

$$g(t) = 1 - (\cos 2t)/2 - 2(\cos 4t)/5 - (\cos 6t)/10 \quad (4.21.28)$$

(cf. [Sh1, p. 80]), and by using (4.18.16) to estimate f_n in (4.21.23) with

sufficient accuracy, the differential equation (4.21.19) becomes transformed again into

$$v_{\tau\tau} + n^2 v = O(1) v \quad (4.21.29)$$

uniformly for θ in (4.21.25) belonging to any fixed closed subinterval of $(0, \pi)$ (cf. [Sh1, p. 83]).

W. C. Bauldry used his Theorem 4.20.4 to obtain equations similar to (4.21.29) for w given by (4.20.29). I am confident that, on the basis of the work [BonCl], one will eventually be able to prove the analogue of (4.21.29) for all weights of the form (4.21.1).

Step (ii). Asymptotics at One Point

This step is easy if the weight function is even (modulo a translation of the variable), whereas otherwise it is associated with one of the most challenging problems in the general theory of orthogonal polynomials which seeks relationships among orthogonal polynomials, measures and three-term linear homogeneous recurrences.

If there is a real b such that $w(b-x) = w(b+x)$ for all real x , then the recurrence coefficient b_n in (3.7) equals b for every n . Thus we have

$$a_{n+1} p_{n+1}(w, b) = -a_n p_{n-1}(w, b), \quad (4.21.30)$$

from which

$$p_{2j+1}(w, b) = 0 \quad (4.21.31)$$

and

$$p_{2j}(w, b) = \gamma_0 (-1)^j \prod_{k=1}^j [a_{2k-1}/a_{2k}] \quad (4.21.32)$$

follow for $j=0, 1, \dots$. Therefore, finding asymptotics for $p_n(w, b)$ can be achieved via asymptotics of the recursive coefficients. If, for instance, w is defined by (4.21.7), then (4.18.16) immediately yields

$$p_n(w, 0) = A \cos(n\pi/2) n^{-1/8} [1 + O(1/n)] \quad (4.21.33)$$

(cf. [Ne31, Lemma 1, p. 1178]).

Now let us examine the general case where in the lack of symmetries, there is no cornerstone to be found where the recurrence formula would become simplified in comparison to the formula taken at an arbitrary point (cf. (3.7) and (4.21.30)). What we need is a method for solving second-order linear (possibly homogeneous) difference equations with variable coefficients. There exist such methods developed by various researchers in connection with investigations regarding continued fractions, discrete scat-

tering theory, Jacobi matrices, orthogonal polynomials, perturbation theory, and so forth (cf. [AgMa, AsIs3, AsWi2, AvSi, Bax1, Bax2, Bes, BeltZu, Bl, Cas1–Cas5, Chi1–Chi8, ChiNe, dBG0, Do1–Do5, DoFr, DoNe, Gaul, Ge1–Ge7, GeCa1, GeCa2, GeNe, GeVa, Ger2, Ger4, Is1–Is5, IsWi, Kre2, Ló2, MáNe3–MáNe5, MáNeTo4, MáNeZa, Ne18, Ne19, Ne22, Ne25, Ne33, Ne35, Ne36, NeDe, Nu1–Nu3, NuSi, Ol, Po, Pol1–Pol3, Rah5, Rah6, Sho1, Va1, Wi1–Wi6, Wils1, Wils2]) but unfortunately (or perhaps luckily), with one exception, none of them suits our purpose since they work only when the corresponding measure has compact support. This exception is my method, which I developed jointly with A. Máté and V. Totik in [MáNe3, MáNeTo4] and which can be described as follows.

For simplicity of presentation, let us assume that we seek asymptotics for $p_n = p_n(0)$ satisfying the recurrence

$$a_{n+1} p_{n+1} + b_n p_n + a_n p_{n-1} = 0. \quad (4.21.34)$$

Let us introduce the characteristic equation of (4.21.34),

$$a_{n+1} t^2 + b_n t + a_n = 0. \quad (4.21.35)$$

The roots of (4.21.35) are

$$t_{jn} = -b_n/(2a_{n+1}) \pm i[a_n/a_{n+1} - b_n^2/(4a_{n+1}^2)]^{1/2}, \quad j = 1, 2. \quad (4.21.36)$$

Define Φ_n by

$$\Phi_n = p_n - t_{1n} p_{n-1}. \quad (4.21.37)$$

Then it can easily be verified that

$$\Phi_{n+1} - t_{2n} \Phi_n = (t_{1,n} - t_{1,n+1}) p_n. \quad (4.21.38)$$

Upon dividing both sides of (4.21.38) by $\prod (t_{2k})$ ($k = 1, 2, \dots, n$) and defining

$$\Psi_n = \Phi_n \prod_{k=1}^{n-1} (t_{2k})^{-1}, \quad (4.21.39)$$

we obtain

$$\Psi_{n+1} = \Psi_n + (t_{1,n} - t_{1,n+1}) p_n \prod_{k=1}^n (t_{2k})^{-1}. \quad (4.21.40)$$

Successive application of (4.21.40) yields

$$\Psi_n = \Psi_1 + \sum_{s=1}^{n-1} (t_{1,s} - t_{1,s+1}) p_s \prod_{k=1}^s (t_{2k})^{-1}. \quad (4.21.41)$$

Now, in (4.21.36), the zeros t_{jn} of the characteristic equation (4.21.35) are wholly determined by the recurrence coefficients a_n and b_n in (4.21.34). Thus any asymptotic expansion of the latter will also result in asymptotics for t_{jn} , in particular for $(t_{1,s} - t_{1,s+1})$ and $\prod (t_{2k})^{-1}$ in (4.21.41).

For weights defined by (4.21.1), Theorem 4.18.4 gives $a_n/a_{n+1} \rightarrow 1$ and $b_n/a_{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $|t_{jn}|^2 = a_n/a_{n+1}$ and

$$|p_n| \leq \text{const } |\Phi_n| \quad (4.21.42)$$

for n sufficiently large (cf. (4.21.37)). Thus (4.21.40) yields

$$|\Psi_n| \leq \text{const } |\Psi_{n-1}| \{1 + |t_{1,n-1} - t_{1,n}|\}, \quad (4.21.43)$$

and, by successive iteration of (4.21.43), we obtain

$$|\Psi_n| \leq \exp \left[\text{const} \left\{ 1 + \sum_{s=1}^{n-1} |t_{1,s} - t_{1,s+1}| \right\} \right]. \quad (4.21.44)$$

If

$$\sum_{s=1}^{\infty} |t_{1,s} - t_{1,s+1}| < \infty, \quad (4.21.45)$$

then one can apply (4.21.41) and (4.21.44) to show

$$\lim_{n \rightarrow \infty} \Psi_n \neq 0 \quad (4.21.46)$$

exists. By (4.21.37) and (4.21.39), we have

$$\text{Im} \{t_{1n}\} p_{n-1} = \text{Im} \left\{ \Psi_n \prod_{k=1}^{n-1} t_{2k} \right\}, \quad (4.21.47)$$

which together with (4.21.46) and asymptotics for the zeros (4.21.36) of the characteristic equation (4.21.35) yields asymptotic estimates for $p_n = p_n(w, 0)$.

These ideas were used by Bauldry in his Ph.D. dissertation [Bau2, Theorem 3.3.2, p. 66] to prove the following

THEOREM 4.21.1 [Bau2]. *Let*

$$w(x) = \exp(-\Pi_4(x)), \quad x \in \mathbb{R}, \quad (4.21.48)$$

where

$$\Pi_4(x) = x^4/4 + q_3 x^3/3 + q_2 x^2/2 + q_1 x, \quad (4.21.49)$$

$q_1, q_2, q_3 \in \mathbb{R}$. Then there exists constants $A > 0$ and c , independent of n , such that

$$p_n(w, 0) = An^{-1/8} \cos[n\pi 2^{-1} - 3^{11/4} 142^{-1} q_1(q_2 - 3) n^{1/4} + c] + o(1) n^{-1/8}. \tag{4.21.50}$$

I point out that although we do not know the exact value of c , as we will see later, A can be determined by a sophisticated method which yields $A = 3^{1/8} \pi^{-1/2}$. I am convinced that the remainder term $o(1)$ in (4.21.50) can be improved to $O(1/n)$. This will require a very careful analysis of all asymptotic formulas involving the recurrence coefficients and the zeros of the characteristic equation (4.21.35).

Step (iii). Plancherel–Rotach Asymptotics

Once we know that the orthogonal polynomials satisfy an equation of the form (4.21.6) such as

$$v_{\tau\tau} + n^2 v = O(1) v, \tag{4.21.51}$$

where v can be expressed in terms of w , $p_n(w)$ and the recurrence coefficients in (3.7), then Plancherel–Rotach-type asymptotics for $p_n(w, x)$ can be obtained by solving (4.21.51) via Liouville–Steklov’s method (cf. [Sz2, p. 210, Ne31, Ne33, Ol]). This requires two pieces of information regarding (4.21.5). First, one has to know the behavior of the initial data at some point, say, asymptotics for $v(0)$ and $v'(0)$. Second, one has to be able to find accurate uniform estimates for v in the intervals where we attempt to find asymptotic solutions of (4.21.51).

All these obstacles can be removed if w is a Freud weight of the form

$$w(x) = \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \tag{4.21.52}$$

where $\Pi_m \in \mathbb{P}_m$. By Theorem 4.20.2, we can write

$$p'_n(w, 0) = \sum_{k=n-m+1}^{n-1} c_{kn} p_k(w, 0), \tag{4.21.53}$$

and thus if we know $p_n(w, 0)$, then we can also determine $p'_n(w, 0)$. For instance, if $w(x) = \exp(-x^6/6)$, then

$$p'_n(w, 0) = p_{n-1}(w, 0) [a_n a_{n+1} (a_{n+2}^2 + a_{n+1}^2 + a_n^2) + a_n^3 (a_{n+1}^2 + a_n^2 + a_{n-1}^2)], \tag{4.21.54}$$

where the a_n ’s are the recursion coefficients in (4.21.24) (cf. [Sh1, p. 30; Sh2]). A method of estimating $p_n(w)$ for weights (4.21.52) was described in

Section 4.18 (cf. Theorem 4.18.13). Once we have asymptotics for $p_n(w, 0)$ and $p'_n(w, 0)$ and estimates for $p_n(w)$, then this can usually be translated into similar asymptotics and estimates for $v(0)$, $v'(0)$ and v , respectively, with no difficulty whatsoever.

There are only three cases where all the details of the above-described analysis have been completed. These are $w(x) = \exp(-x^2)$ (Hermite polynomials, by M. Plancherel and W. Rotach [PIRo]; cf. [Sz2, p. 200]), $w(x) = \exp(-x^4)$ (by me in [Ne31]) and $w(x) = \exp(-x^6)$ (by R. Sheen [Sh1, Sh2]). For example, Ron Sheen's asymptotic formula is given by

THEOREM 4.21.2 [Sh2]. *Let $w(x) = \exp(-x^6/6)$, $x \in \mathbb{R}$, $0 < \varepsilon < \pi/2$ and $x = (32n/5)^{1/6} \cos \theta$. Then*

$$\begin{aligned} \exp(-x^6/12) p_n(w, x) &= 10^{1/12} \pi^{-1/2} n^{-1/12} (\sin \theta)^{-1/2} \\ &\times \cos[n60^{-1}(60\theta - 15 \sin 2\theta - 6 \sin 4\theta - \sin 6\theta) + \theta 2^{-1} - \pi 4^{-1}] \\ &+ O(n^{-13/12}), \end{aligned} \tag{4.21.55}$$

uniformly for $n = 1, 2, \dots$ and $\varepsilon \leq \theta \leq \pi - \varepsilon$.

Although the final product is smooth and polished, there is one problem intrinsic to the nature of my method. Namely, one has to prove (4.21.55) with $x=0$ before one can proceed with the general case. However, for $x=0$, one can only prove (4.21.55) with some constant $A > 0$ instead of $10^{1/12} \pi^{-1/2}$. Therefore, initially one proves a weaker version of (4.21.55) where $10^{1/12} \pi^{-1/2}$ is replaced by A . The determination of the value of the constant A is then achieved by showing that

$$\lim_{a \rightarrow 1} \lim_{n \rightarrow \infty} \int_{-a(32n/5)^{1/6}}^{a(32n/5)^{1/6}} p_n(w, x)^2 w(x) dx = 1 \tag{4.21.56}$$

(cf. [Ne31, p. 1184]) and by substituting the asymptotic formula (4.21.55) for the integrand in (4.21.56), which can be justified since the asymptotics is valid between the limits of integration.

For nonsymmetric weight functions there is only one partial result (containing a nondetermined constant) by Bill Bauldry [Bau2, Theorem 4.1], who treated the weight function $w(x) = \exp(-\Pi_4(x))$.

In my survey papers [Ne33, Ne36], I proposed the following conjecture.

Conjecture 4.21.3. Let

$$w(x) = \exp(-|x|^m), \quad x \in \mathbb{R}, \quad m > 1, \tag{4.21.57}$$

and let $0 < \varepsilon < \pi/2$. Then the asymptotic formula

$$\begin{aligned} \exp(-|x|^{m/2}) p_n(w, x) &= [\Gamma(m+1) \Gamma(m/2)^{-1} \Gamma((m/2)+1)^{-1}]^{1/2m} \pi^{-1/2} n^{-1/2} n^{-1/2m} (\sin \theta)^{-1/2} \\ &\times \cos \left[n(\theta - \text{sign}(\cos \theta) |\cos \theta|^m) \int_{|\cos \theta|}^1 t^{-m} (1-t^2)^{-1/2} dt + \theta/2 - \pi/4 \right] \\ &+ O(n^{-1/(2m)-1}) \end{aligned} \quad (4.21.58)$$

holds uniformly for $n = 1, 2, \dots$ and $\varepsilon \leq \theta \leq \pi - \varepsilon$, where

$$x = [\pi^{1/2} n \Gamma(m/2) \Gamma((m+1)/2)^{-1}]^{1/m} \cos \theta. \quad (4.21.59)$$

J. Nuttall pointed out to me that, in the original Plancherel–Rotach asymptotics for the Hermite polynomials, x is given by

$$x = (2n+1)^{1/2} \cos \theta \quad (4.21.60)$$

(cf. [Sz2, Theorem 8.22.9, p. 201]) whereas (4.21.59), with $m=2$, yields

$$x = (2n)^{1/2} \cos \theta. \quad (4.21.61)$$

The different parametrization of x in (4.21.60) and (4.21.61) accounts for the slight discrepancy between the case $m=2$ in (4.21.58) and the Plancherel–Rotach asymptotic expansion (8.22.12) in [Sz2, p. 201].

Although it might take a long time to prove (4.21.58) in its entire generality, I have good reasons for and faith in believing that the case where m is an even integer in (4.21.57) will soon be taken care of.

4.22. *Plancherel–Rotach Asymptotics for Christoffel Functions with Freud Weights*

Well, my reader, rejoice. This is the climactic convergence of ideas presented in the Thesis in Section 2 and analyzed in Sections 4.1–4.21. It brings together the main subjects/objects of my study in a way that would have indeed pleased Freud had he been fortunate to live long enough to see such asymptotics.

By the Christoffel–Darboux formula (3.13), applied with $x=t$, we have

$$\lambda_n(w, x)^{-1} = a_n(w) [p'_n(w, x) p_{n-1}(w, x) - p_n(w, x) p'_{n-1}(w, x)], \quad (4.22.1)$$

where $a_n(w)$ is the recurrence coefficient in (3.7). Therefore asymptotics for $a_n(w)$, $p_n(w, x)$ and $p'_n(w, x)$ leads to asymptotics for the Christoffel functions $\lambda_n(w, x)$. For the recurrence coefficients $a_n(w)$ we have results mentioned in Section 4.18 such as Theorem 4.18.5, and for $p_n(w, x)$ some asymptotics and a conjecture were discussed in Section 4.21. Although

differentiation of asymptotic formulas is usually very difficult, if not impossible, to justify, there is a case where it is easy to do, at least in principle. This is the case where the derivatives of orthogonal polynomials are quasi-orthogonal in the sense of (4.20.16). Hence, by Theorem 4.20.2, it is exactly the Freud weights of the form

$$w(x) = \exp(-\Pi_m(x)), \quad x \in \mathbb{R}, \tag{4.22.2}$$

where Π_m is a polynomial of degree m with positive leading coefficient, when one should be able to carry out the necessary computations leading to Plancherel–Rotach-type asymptotics for Christoffel functions.

There are only two weight functions for which all the details of the above-described analysis have been completed. These are $w(x) = \exp(-x^4)$ (by me in [Ne31, Theorem 2, p. 1178]) and $w(x) = \exp(-x^6)$ (by R. C. Sheen in [Sh1, Theorem 3.2, p. 63; Sh2]). Of course, the Christoffel functions of the Hermite weight $w(x) = \exp(-x^2)$ can easily be taken care of in view of Plancherel and Rotach’s asymptotic formula [PIRo] (cf. [Sz2, p. 201]) and (4.20.13). For comparison with Theorem 4.21.2, I give Sheen’s

THEOREM 4.22.1 [Sh2]. *Let $w(x) = \exp(-x^6/6)$, $x \in \mathbb{R}$, $0 < \varepsilon < \pi/2$ and $x = (32n/5)^{1/6} \cos \theta$. Then*

$$\begin{aligned} n^{-5/6} \exp(-x^6/6) \lambda_n(w, x)^{-1} \\ = 10^{-5/6} \pi^{-1} \sin \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) + O(n^{-1}), \end{aligned} \tag{4.22.3}$$

uniformly for $n = 1, 2, \dots$ and $\varepsilon \leq \theta \leq \pi - \varepsilon$.

The transition from Theorem 4.21.2 to Theorem 4.22.1 is accomplished via the identity

$$\begin{aligned} \lambda_n(w, x)^{-1} = a_n^2 \varphi_{n-1}(x) p_n(w, x)^2 + a_n^2 \varphi_n(x) p_{n-1}(w, x)^2 \\ + a_n [\delta_{n-1}(x) - \delta_n(x) - x \varphi_{n-1}(x)] p_n(w, x) p_{n-1}(w, x), \end{aligned} \tag{4.22.4}$$

where

$$\begin{aligned} \varphi_n(x) = a_{n+1}^2 (a_{n+2}^2 + a_{n+1}^2 + a_n^2) + a_n^2 (a_{n+1}^2 + a_n^2 + a_{n-1}^2) \\ + x^2 (a_{n+1}^2 + a_n^2 + x^2) \end{aligned} \tag{4.22.5}$$

and

$$\delta_n(x) = x a_n^2 (a_{n+1}^2 + a_n^2 + a_{n-1}^2 + x^2) \tag{4.22.6}$$

(there the a_n ’s are the recursion coefficients which satisfy (4.21.24)).

I humbly admit that the number of rigorously proved results here is fairly moderate compared to what I expect to emerge in the near future.

Moreover, it is somewhat embarrassing that aesthetically pleasing theorems are accompanied by proofs in which splendid ideas are combined with occasional computations capable of provoking the reader's patience.

Therefore it is fair to conclude this work with the following conjecture, whose beauty is unquestionable.

Conjecture 4.22.2 [Ne36]. Let

$$w(x) = \exp(-|x|^m), \quad x \in \mathbb{R}, \quad m > 1, \quad (4.22.7)$$

and let $0 < \varepsilon < \pi/2$. Let

$$x = [\pi^{1/2} n \Gamma(m/2) \Gamma((m+1)/2)^{-1}]^{1/m} \cos \theta. \quad (4.22.8)$$

Then the asymptotic formula

$$\begin{aligned} n^{-1/m-1} \exp(-|x|^m) \lambda_n(w, x)^{-1} = \\ m[\Gamma(m+1) \Gamma(m/2)^{-1} \Gamma((m/2)+1)^{-1}]^{1/m} \pi^{-1} \\ \times |\cos \theta|^{m-1} \int_{|\cos \theta|}^1 t^{-m} (1-t^2)^{-1/2} dt + O(n^{-1}) \end{aligned} \quad (4.22.9)$$

holds uniformly for $n = 1, 2, \dots$ and $\varepsilon \leq \theta \leq \pi - \varepsilon$.

5. EPILOGUE

Yes, my reader, I owe you a confession and beg for your generous forgiveness. Having read Section 4, you must have observed that I deceived you when I promised in the Thesis (cf. Section 2) to spend the rest of this essay praising Géza Freud and his contributions to the theory of orthogonal polynomials. Instead, I ended up criticizing my former advisor, friend and mentor for not accomplishing what eventually has been conjectured, formulated, nourished and proved by a new generation of enthusiastic experts on orthogonal polynomials.

On the other hand, you must have observed as well that I faithfully followed up my pledge to dig to the roots of Freud's devotion to orthogonal polynomials and Christoffel functions, and that I analyzed the circumstances that were behind his endeavor to apply Christoffel functions to almost all problems in orthogonal polynomials that his hands ever touched.

In the Thesis I formulated Freud's five major contributions to orthogonal polynomials, namely his work on (i) Tauberian theorems; (ii) Cesàro summability of orthogonal Fourier series; (iii) asymptotics for orthogonal polynomials; (iv) convergence of orthogonal Fourier series, interpolation processes, and quadrature sums; and (v) orthogonal

polynomials associated with exponential weights on infinite intervals. What I did not mention there, and what is perhaps even more significant, is that it was precisely Freud whose fervent research covering a quarter of a century provided continuity in the development of the general theory of orthogonal polynomials which in the first half of this century vigorously flourished in the works of N. I. Akhiezer, S. N. Bernstein, P. Erdős, L. Fejér, Ya. L. Geronimus, A. N. Kolmogorov, M. G. Krein, M. Riesz, J. A. Shohat, V. I. Smirnov, P. Turán, and G. Szegő. It was Freud who kept the ashes in one pile so that the phoenix of orthogonal polynomials could rise again and enjoy an ever increasing popularity which a generation ago would have been inconceivable.

ACKNOWLEDGMENTS

I have entertained the idea of analyzing Géza and his research since his untimely death on September 27, 1979. My original plan was to write a treatise on his work in approximation theory and orthogonal polynomials. I soon realized the immense nature of such an attempt, and thus I chose to limit myself to Géza's research on orthogonal polynomials, while I asked others to report on his contributions to various areas of approximation theory (cf. the first issue in volume 46 of this Journal).

The contents of this work were continually discussed with Vili Totik. I express my genuine appreciation to Vili for his patience and reassuring encouragement.

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Needless to say, the judgments, opinions, speculations and views expressed in this essay are strictly mine and I assume full responsibility for them.

Note added in proof. As it was to be expected, research in orthogonal polynomials did not cease after the completion of this work. As a matter of fact, there has been a great deal of renewed interest in the subject which I hope was partially spurred by the widespread distribution of preprints of this essay. Among the many important recent results, I mention here only one which I strongly believe is the most significant breakthrough: Freud's Conjecture 4.18.1 regarding the recursion coefficients of orthogonal polynomials associated with exponential weights has been proved by D. S. Lubinsky, H. N. Mhaskar and E. B. Saff.

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